SIAM J. MATH. ANAL.
Vol. 57, No. 2, pp. 1964–1989

TIME-HARMONIC ELASTIC SCATTERING BY UNBOUNDED DETERMINISTIC AND RANDOM ROUGH SURFACES IN THREE DIMENSIONS*

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Abstract. In this paper, we investigate well-posedness of time-harmonic scattering of elastic waves by unbounded rigid rough surfaces in three dimensions. Both deterministic and random surfaces are investigated through the variational approach. In a deterministic setting, the rough surface is assumed to be a Lipschitz profile, which lies within a finite distance of a flat plane. The elastic scattering is caused by a source term with a compact support above the rough surface. A stability estimate is derived at an arbitrary frequency, with explicit dependance of the bound on the excitation frequency and geometry of rough surfaces. Based on the a priori dependence on the frequency together with the measurability and P-essentially separability of the randomness, we obtain a similar bound for the time-harmonic elastic scattering by random surfaces.

Key words. elastic waves, rough surfaces, variational formulation, explicit a priori bounds

MSC codes. 35A15, 35P25, 74J20

DOI. 10.1137/24M1640306

1. Introduction.

This paper is concerned with the mathematical analysis of time-harmonic elastic scattering from unbounded deterministic and random rough surfaces in three dimensions. The phrase rough surface means a (usually nonlocal) perturbation of an infinite plane such that the whole surface lies within a finite distance of the original plane. Rough surface scattering problems have important applications in diverse scientific areas such as remote sensing, geophysics, outdoor sound propagation, and radar techniques (see e.g.,[1, 2] and the references cited therein). In linear elasticity, most well-posedness results are confined to two-dimensional settings and periodic structures. Existence and uniqueness of solutions were studied via the boundary integral equation method [3, 4, 5] based on the upward propagation radiation condition. The variational approach was proposed in [9, 11] to treat well-posedenss of the scattering problems in periodic structures by using the Rayleigh expansion condition and in [10, 12] for rigid rough surfaces by using the angular spectrum representation (ASR).

Recently, the ASR of solutions to the homogeneous Navier equation in a half-space was presented in three dimensions in [13]. Based on a Rellich-type identity, uniqueness

*Received by the editors February 20, 2024; accepted for publication (in revised form) January 7, 2025; published electronically April 21, 2025.
https://doi.org/10.1137/24M1640306
Funding: The research of GH is supported in part by NSFC (12425112 and 12071236). The research of XX is supported in part by National Key Research and Development Program of China (2024YFA1012300, 2023YFA1009100) and NSFC (12071430), Key Laboratory of Collaborative Sensing and Autonomous Unmanned Systems of Zhejiang Province, and the Open Research Project of Innovation Center of Yangtze River Delta, Zhejiang University. The research of YZ is supported in part by NSFC (12371421).
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of weak solutions to the variational problem was verified provided the rigid scattering surface was the graph of a uniformly Lipschitz continuous function. Note that the uniqueness proof implies absence of surface waves propagating along the rough surface, due to the geometric assumption on the scattering surface. The existence of solutions was proved only for the case of locally perturbed scattering problems in [13]. However, the well-posedness of elastic scattering by nonlocally perturbed and nonperiodic rough surfaces remains unknown in three dimenions. Later, the authors in [16] further derived an a priori bound for the two-dimensional problem, which depends explicitly on the excitation frequency. The main goal of this paper is threefold. First, we present a variational formulation of the elastic scattering problems with a Lipschitz-type rough surface in three dimensions and prove its well-posedness if the source term is compactly supported. Second, we derive an a priori bound with explicit dependence on the excitation frequency. Third, we utilize this explicit bound to derive the well-posedness of elastic scattering by random rough surfaces in three dimensions, motivated by the analogous results in [6, 16] in two dimensions. As discussed in [7], we expect that the variational formulation will be suitable for numerical approximations in nonsmooth and inhomogeneous domains via the finite element discretization. Furthermore, the explicit bounds obtained in this paper should be useful in establishing the dependence of the constants in a priori error estimates for finite element schemes on the frequency and the geometry of the domain.

This paper extends the existing variational method of [10, 16] to three-dimensional settings and complements the well-posedness result in [13] for deterministic surfaces. As was pointed out in [10], the elastic problem is more complicated than the acoustic case due to the coexistence of compressional and shear waves. As a consequence, the Dirichlet-to-Neumann map for the elastic wave equation is defined in the space of vector-valued functions and the real part of this map is not positively definite. This brings difficulties in deriving the a priori estimates of solutions via Rellich identities for arbitrary frequencies. We prove that the variational problem is well-posed by the theory of semi-Fredholm operators used in [10]. To this end, we first consider the case of small frequencies in which the Lax–Milgram theorem can be applied. Then we establish several a priori estimates. During this process, we carefully trace the dependence of the coefficients of these bounds on the frequency. In this way, we arrive at an a priori bound for the solution to the variational problem which is explicitly dependent on the frequency. Afterwords, inspired by the framework for scattering by random meda in [15] and random surfaces in [6, 16], we can obtain the well-posedness for a stochastic variation problem with an explicit a priori bound.

The rest of this paper is outlined as follows. In section 2 we present the variational formulation for the elastic scattering problem. Section 3.3 is devoted to the well-posedness of the variational problem for small frequencies. In section 4 we derive a priori bounds and trace the explicit dependence on the frequency and on the geometry of the domain. For random cases, a similar bound is derived in section 5. Conclusions are presented in section 6.

2. Problem formulation.

This section is devoted to the mathematical formulation of the three-dimensional elastic wave scattering by unbounded rigid rough surfaces. Let $D \subset \mathbb{R}^3$ be an unbounded connected open set such that, for some constants $m < M$,

$$U_M \subset D \subset U_m, \quad U_h := \{x = (x', x_3) : x_3 > h\}, \quad x' := (x_1, x_2). \tag{2.1}$$

The region D is supposed to be filled with a homogeneous and isotropic elastic medium with unit mass density. We assume that $\Gamma := \partial D$ is an unbounded rough surface, which is the graph of a uniformly Lipschitz continuous function f. More precisely, we assume

$$\Gamma = \{x \in \mathbb{R}^3 : x_3 = f(x'),\, x' = (x_1, x_2) \in \mathbb{R}^2\},$$

and there exists a constant $L > 0$ such that

$$|f(x') - f(y')| \le L\,|x' - y'| \quad \text{for all} \quad x', y' \in \mathbb{R}^2. \tag{2.2}$$

Throughout the paper we fix some $h > M$. Let $\Gamma_h = \{x \in \mathbb{R}^3 : x_3 = h\}$ and $S_h = D \backslash \overline{U}_h$. Denote the unit normal vector on $\Gamma \cup \Gamma_h$ by $\nu := (\nu_1, \nu_2, \nu_3)$ pointing into the region of $x_3 > h$ on Γ_h and into the exterior of D on Γ. Assume that $g \in L^2(D)^3$ is an elastic source term with $\mathrm{supp}(g) \subset S_h$. Consider the following Navier equation in three dimensions,

$$\Delta^* u + \omega^2 u = g \quad \text{in } D, \tag{2.3}$$

where $\Delta^* = \mu\Delta + (\lambda + \mu)\nabla\nabla\cdot$, $u = (u_1, u_2, u_3)^\top$ is the elastic displacement, and $\omega > 0$ is the angular frequency. Here λ and μ denote the Lamé constants characterizing the medium above Γ satisfying $\mu > 0, \lambda + 2\mu/3 > 0$. Since Γ is physically rigid, there holds the Dirichlet boundary condition

$$u = 0 \quad \text{on } \Gamma. \tag{2.4}$$

As the domain D is unbounded, a proper radiation condition should be imposed on u at infinity. In this paper we utilize the elastic upward ASR (UASR) at infinity to ensure the well-posedness of the boundary value problem (2.3)–(2.4). Below we briefly introduce this radiation condition and refer to [10, 13] for the details. We begin with the decomposition of the wave fields into a sum of compressional and shear parts (see [14]),

$$u = \frac{1}{\mathrm{i}}(\nabla\varphi + \nabla \times \psi), \quad \nabla \cdot \psi = 0 \quad \text{in} \quad x_3 > h, \tag{2.5}$$

where the scalar function φ and the vector function ψ satisfy the homogeneous Helmholtz equations

$$\Delta\varphi + k_{\mathrm{p}}^2\varphi = 0, \quad \Delta\psi + k_{\mathrm{s}}^2\psi = 0 \quad \text{in} \quad x_3 > h. \tag{2.6}$$

Here, k_{p} and k_{s} are compressional and shear wave numbers, respectively, defined by

$$k_{\mathrm{p}} := \frac{\omega}{\sqrt{\lambda + 2\mu}}, \quad k_{\mathrm{s}} := \frac{\omega}{\sqrt{\mu}}. \tag{2.7}$$

Denote by $\hat{v}$ the Fourier transform of v in $\mathbb{R}^2$, i.e.,

$$\hat{v}(\xi) = \mathcal{F}v(\xi) := \frac{1}{2\pi}\int_{\mathbb{R}^2} v(x')e^{-\mathrm{i}x'\cdot\xi}\,\mathrm{d}x', \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Taking the Fourier transform of (2.6) and assuming that φ, ψ fulfill the UASR of the Helmholtz equation in U_h (see [7]), we obtain for $x_3 \ge h$ that

$$\varphi(x', x_3) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \hat{\varphi}(\xi, h) e^{\mathrm{i}\beta(\xi)(x_3-h)} e^{\mathrm{i}\xi\cdot x'} \mathrm{d}\xi,$$

$$\psi(x', x_3) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \hat{\psi}(\xi, h) e^{\mathrm{i}\gamma(\xi)(x_3-h)} e^{\mathrm{i}\xi\cdot x'} \mathrm{d}\xi, \tag{2.8}$$

where

$$\beta(\xi) := \begin{cases} (k_{\mathrm{p}}^2 - |\xi|^2)^{1/2}, & |\xi| < k_{\mathrm{p}}, \\ \mathrm{i}(|\xi|^2 - k_{\mathrm{p}}^2)^{1/2}, & |\xi| > k_{\mathrm{p}}, \end{cases}$$

and

$$\gamma(\xi) := \begin{cases} (k_{\mathrm{s}}^2 - |\xi|^2)^{1/2}, & |\xi| < k_{\mathrm{s}}, \\ \mathrm{i}(|\xi|^2 - k_{\mathrm{s}}^2)^{1/2}, & |\xi| > k_{\mathrm{s}}. \end{cases}$$

Denote the Fourier transform of $\varphi(x', h)$ and $\psi(x', h)$ by

$$A_{\mathrm{p}}(\xi) = \hat{\varphi}(\xi, h), \quad \tilde{\boldsymbol{A}}_{\mathrm{s}}(\xi) = \hat{\psi}(\xi, h),$$

respectively. Noting that $\operatorname{div}\psi = 0$, we have $(\xi, \gamma(\xi)) \cdot \tilde{\boldsymbol{A}}_{\mathrm{s}}(\xi)^\top = 0$. For notational convenience we omit the dependence of β and γ on ξ in the subsequent context.

Substituting (2.8) into (2.5), we obtain for $x_3 \geq h$ that

$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} [A_{\mathrm{p}}(\xi)\,(\xi, \beta)^\top e^{\mathrm{i}\beta(x_3-h)} + \boldsymbol{A}_{\mathrm{s}}(\xi)\, e^{\mathrm{i}\gamma(x_3-h)}] e^{\mathrm{i}\xi\cdot x'} \mathrm{d}\xi, \tag{2.9}$$

where $\boldsymbol{A}_{\mathrm{s}} = (A_{\mathrm{s}}^{(1)}, A_{\mathrm{s}}^{(2)}, A_{\mathrm{s}}^{(3)})^\top(\xi) := (\xi, \gamma)^\top \times \tilde{\boldsymbol{A}}_{\mathrm{s}}(\xi)$. It follows from (2.9) and the orthogonality $(\xi, \gamma) \cdot \boldsymbol{A}_{\mathrm{s}}^\top = 0$ that

$$\begin{bmatrix} \hat{u}(\xi, h) \\ 0 \end{bmatrix} = \begin{bmatrix} \xi_1 & 1 & 0 & 0 \\ \xi_2 & 0 & 1 & 0 \\ \beta & 0 & 0 & 1 \\ 0 & \xi_1 & \xi_2 & \gamma \end{bmatrix} \begin{bmatrix} A_{\mathrm{p}}(\xi) \\ \boldsymbol{A}_{\mathrm{s}}(\xi) \end{bmatrix} := \widetilde{\mathbb{D}}(\xi)\, \boldsymbol{A}(\xi),$$

which gives

$$\boldsymbol{A}(\xi) = \begin{bmatrix} A_{\mathrm{p}} \\ \boldsymbol{A}_{\mathrm{s}} \end{bmatrix}(\xi) = \widetilde{\mathbb{D}}^{-1}(\xi) \begin{bmatrix} \hat{u}(\xi, h) \\ 0 \end{bmatrix} = \mathbb{D}(\xi)\, \hat{u}(\xi, h). \tag{2.10}$$

Here $\mathbb{D}$ is a 4×3 matrix given by

$$\mathbb{D}(\xi) = \frac{1}{\beta\gamma + |\xi|^2} \begin{bmatrix} \xi_1 & \xi_2 & \gamma \\ \beta\gamma + \xi_2^2 & -\xi_1\xi_2 & -\xi_1\gamma \\ -\xi_1\xi_2 & \beta\gamma + \xi_1^2 & -\xi_2\gamma \\ -\xi_1\beta & -\xi_2\beta & |\xi|^2 \end{bmatrix}. \tag{2.11}$$

Using (2.9)–(2.10) yields the expression of u in U_h,

$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \left\{ \frac{1}{\beta\,\gamma + |\xi|^2} (M_{\mathrm{p}}(\xi) e^{\mathrm{i}(\xi\cdot x' + \beta(x_3-h))} + M_{\mathrm{s}}(\xi) e^{\mathrm{i}(\xi\cdot x' + \gamma(x_3-h))}) \hat{u}(\xi, h) \right\} \mathrm{d}\xi, \tag{2.12}$$

where

$$M_{\mathrm{p}}(\xi) =: \begin{bmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\gamma \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\gamma \\ \xi_1\beta & \xi_2\beta & \beta\gamma \end{bmatrix} \text{ and } M_{\mathrm{s}}(\xi) = \begin{bmatrix} \beta\gamma + \xi_2^2 & -\xi_1\xi_2 & -\gamma\xi_1 \\ -\xi_1\xi_2 & \beta\gamma + \xi_1^2 & -\gamma\xi_2 \\ -\xi_1\beta & -\xi_2\beta & |\xi|^2 \end{bmatrix}.$$

The representation (2.12) will be referred to as the upward radiation condition for rough surface scattering problems in linear elasticity.

Define the surface traction operator

$$Tu := 2\mu\partial_\nu u + \lambda(\nabla\cdot u)\nu + \mu\nu\times(\nabla\times u), \tag{2.13}$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ stands for the normal vector on the surface. Plugging (2.12) into (2.13) yields the Dirichlet-to-Neumann (DtN) operator on Γ_h (cf. [13])

$$Tu = \mathcal{T}u(x') := \frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}^2}\mathcal{M}(\xi)\hat{u}(\xi)e^{\mathrm{i}\xi\cdot x'}\mathrm{d}\xi, \tag{2.14}$$

where $\mathcal{M}(\xi)$ is given by

$$\begin{aligned}\mathcal{M}(\xi) &= \frac{1}{|\xi|^2+\beta\gamma}\\ &\times\begin{bmatrix}\mu[(\gamma-\beta)\xi_2^2+k_s^2\beta] & -\mu\xi_1\xi_2(\gamma-\beta) & (2\mu|\xi|^2-\omega^2+2\mu\beta\gamma)\xi_1\\ -\mu\xi_1\xi_2(\gamma-\beta) & \mu[(\gamma-\beta)\xi_1^2+k_s^2\beta] & (2\mu|\xi|^2-\omega^2+2\mu\beta\gamma)\xi_2\\ -(2\mu|\xi|^2-\omega^2+2\mu\beta\gamma)\xi_1 & -(2\mu|\xi|^2-\omega^2+2\mu\beta\gamma)\xi_2 & \gamma\omega^2\end{bmatrix}.\end{aligned} \tag{2.15}$$

The boundary operator $\mathcal{T}$ is nonlocal and $\mathcal{T}u = Tu$ is equivalent to the upward radiation condition (2.12). It is also called the transparent boundary condition for time-harmonic scattering problems in a half-space.

Based on the above DtN operator, the wave scattering problem (2.3)–(2.4) can be reduced to a boundary value problem over S_h:

$$\begin{aligned}\mu\Delta u + (\lambda+\mu)\nabla\nabla\cdot u + \omega^2 u = g &\quad \text{in } S_h,\\ u = 0 &\quad \text{on } \Gamma,\\ Tu = \mathcal{T}u &\quad \text{on } \Gamma_h.\end{aligned}$$

To introduce the variational formulation, we introduce the energy space V_h for $h > M$ as the closure of $C_0^\infty(S_h\cup\Gamma_h)^3 := \{u|_{S_h} : u\in C_0^\infty(D)^3\}$ in the H^1 norm

$$\|u\|_{V_h} = (\|\nabla u\|^2_{L^2(S_h)^3} + \|u\|^2_{L^2(S_h)^3})^{1/2}.$$

Multiplying the Navier equation in (2.3) by the complex conjugate of the test function $v\in V_h$ and using Betti's formula yield

$$\int_{S_h}\mathcal{E}(u,\bar{v}) - \omega^2 u\cdot\bar{v}\,\mathrm{d}x - \int_{\Gamma_h}\bar{v}\cdot Tu\mathrm{d}s = -\int_{S_h} g\cdot\bar{v}\,\mathrm{d}x,$$

where the bilinear form $\mathcal{E}(\cdot,\cdot)$ is defined by

$$\mathcal{E}(u,v) := 2\mu\sum_{j,k=1}^{3}\partial_k u_j\partial_k v_j + \lambda\nabla\cdot u\nabla\cdot v - \mu\nabla\times u\cdot\nabla\times v \quad \forall u,v\in V_h.$$

Define the sesquilinear form $B : V_h\times V_h\to\mathbb{C}$ by

$$B(u,v) = \int_{S_h}\mathcal{E}(u,\bar{v}) - \omega^2 u\cdot\bar{v}\,\mathrm{d}x - \int_{\Gamma_h}\bar{v}\cdot\mathcal{T}u\mathrm{d}s. \tag{2.16}$$

Now we can formulate the variational problem as follows:

Variational problem I*:* find $u\in V_h$ such that

$$B(u,v) = -\int_{S_h} g\cdot\bar{v}\,\mathrm{d}x \quad \text{for all } v\in V_h. \tag{2.17}$$

The variational problem is equivalent to the boundary value problem: given $g \in L^2(D)^3$, with $\mathrm{supp}(g) \subset S_h$ for some $h > M$, find $u \in H^1_{loc}(D)^3$ such that $u|_{S_h} \in V_h$ for every $h > M$ (implying $u = 0$ on Γ), the Navier equation $(\Delta^* + \omega^2)\, u = g$ in D holds in a distributional sense, and the radiation condition (2.12) is satisfied with $u|_{\Gamma_h} \in H^{1/2}(\Gamma_h)^3$ by the trace theorem.

The main theorem of this paper can now be stated as follows.

THEOREM 2.1. *For any $\omega > 0$, the variational problem* I (2.17) *is uniquely solvable in V_h. Moreover, there exists a constant C independent of the frequency ω, the truncation height h, the Lipschitz constant L of f, and the constant m given in* (2.1) *such that the solution satisfies the estimate*

$$\|u\|_{V_h} \le (h - m + 2)(C_4(\omega,h) + C_5(\omega,h)^2 + C_6(\omega,h,L))\,\|g\|_{V_h}, \tag{2.18}$$

where

$$C_4(\omega,h) = C(h+1-m)\omega, \quad C_5 = C\sqrt{1+\omega^{-1}}C_3(\omega,h),$$

and

$$C_6 = C(\omega^{-1}+1)C_1(\omega,h,L)C_2(\omega,h,L)^2.$$

Here

$$\begin{aligned}
C_1(\omega,h,L) &= C\omega^3(1+L^2)^{1/2}(h-m+1),\\
C_2(\omega,h,L) &= C(1+L^2)^{1/4}\sqrt{h+1-m}(1+\omega(h+1-m)),\\
C_3(\omega,h) &= C(h+1-m)(1+\omega(h+1-m))^2/\omega.
\end{aligned}$$

The constants C_1–C_6 are derived from a priori bounds of the variational solution, which exhibit explicit dependence on the frequency ω and the geometry of the rough surface. They lead to the explicit a priori bound of the solution of the elastic scattering problem in three dimensions.

By the theory on semi-Fredholm operators in [10], the results of Theorem 2.1 follow from the well-posedness of the variational problem at small frequencies (cf. Theorem 3.3) and an a priori bound of the solution to the variational problem at an arbitrary frequency (cf. Theorem 4.3). Thus, in the subsequent two sections we shall focus on mathematical analysis at small frequencies and an a priori estimate at an arbitrary frequency.

3. Analysis of the variational problem for small frequency.

We first investigate mapping properties of the DtN operator in three dimensions. For a matrix $\mathcal{M}(\xi) \in \mathbb{C}^{3\times 3}$ depending on ξ, let $\mathrm{Re}\mathcal{M}(\xi) := (\mathcal{M}(\xi) + \mathcal{M}(\xi)^*)/2$. We shall write $\mathrm{Re}\mathcal{M}(\xi) > 0$ if $\mathrm{Re}\mathcal{M}(\xi)$ is positive-definite. Here $\mathcal{M}^*(\xi)$ is the adjoint of $\mathcal{M}$ with respect to the scalar product $(\cdot,\cdot)_{\mathbb{C}^{3\times 3}}$ in $\mathbb{C}^{3\times 3}$.

LEMMA 3.1. *Let $\mathcal{M}(\xi)$ be defined in* (2.15) *and let $h > M$.*

1. *There exists a constant K independent of ω such that $\Re(-i\mathcal{M})(\xi) > 0$ for all $|\xi| > K\omega$, where*

$$K = \frac{\lambda + 2\mu}{\mu\sqrt{\lambda+\mu}} > \frac{1}{\sqrt{\mu}}.$$

2. *The DtN map $\mathcal{T}$ is a bounded operator from $H^{1/2}(\Gamma_h)^3$ to $H^{-1/2}(\Gamma_h)^3$.*

3. *For $|\xi| < K\omega$ there holds that*

$$\|\mathcal{M}(\xi)\| \le C_K\,\omega, \tag{3.1}$$

where

$$C_K = 2(\lambda+4\mu)K + (\mu(\lambda+2\mu)K^2 + 2(\lambda+2\mu))\sqrt{\frac{\lambda+\mu}{\mu(\lambda+2\mu)}}$$

is a constant independent of ω and ξ with the matrix norm defined by

$$\|\mathcal{M}(\xi)\| = \max_{1\le i,j\le 3} |M_{ij}(\xi)|.$$

Here K is the constant specified in item 1 and $M_{ij}(\xi), 1 \le i,j \le 3$, denote the entries of $\mathcal{M}(\xi)$.

Remark 3.2. In comparison with properties of the matrix $\mathcal{M}$ in two dimensions, we provide explicit constants K and C_K in terms of the Lamé coefficients.

Proof. Item 2 has been proved in [13, Lemma 3.2]. Thus we only need to prove items 1 and 3.

(1) Since $|\xi| > K\omega > k_s$, we have $\beta = \mathrm{i}|\beta|$ and $\gamma = \mathrm{i}|\gamma|$, which imply

$$i\mathcal{M}(\xi) = \frac{-1}{|\xi|^2 - |\beta||\gamma|}\begin{bmatrix} a_1(\xi) & b(\xi) & -ic(\xi)\xi_1 \\ b(\xi) & a_2(\xi) & -ic(\xi)\xi_2 \\ ic(\xi)\xi_1 & ic(\xi)\xi_2 & a_3(\xi)\end{bmatrix} := \frac{-1}{|\xi|^2 - |\beta||\gamma|}\mathcal{M}_1(\xi) \tag{3.2}$$

with

$$a_1(\xi) = \mu[\xi_2^2(|\gamma|-|\beta|) + k_s^2|\beta|], \quad a_2(\xi) = \mu[\xi_1^2(|\gamma|-|\beta|) + k_s^2|\beta|], \quad a_3(\xi) = \omega^2|\gamma|,$$
$$b(\xi) = -\mu\xi_1\xi_2(|\gamma|-|\beta|), \quad c(\xi) = 2\mu|\xi|^2 - \omega^2 - 2\mu|\beta||\gamma|.$$

It is obvious that $a_i(\xi)$, $b(\xi)$, $c(\xi) \in \mathbb{R}$. Then from (3.2) we obtain

$$\Re(-\mathrm{i}\mathcal{M}(\xi)) = \frac{1}{|\rho|}\mathcal{M}_1(\xi)$$

with $\rho(\xi) = |\xi|^2 + \beta\gamma$. Hence it remains to prove that $\mathcal{M}_1(\xi)$ is positive-definite if $|\xi| > K\omega$. To this end, we should verify

$$\text{i)}\, a_1(\xi) > 0, \quad \text{ii)} \begin{vmatrix} a_1(\xi) & b(\xi) \\ b(\xi) & a_2(\xi)\end{vmatrix} > 0, \quad \text{iii)}\ \det \mathcal{M}_1(\xi) > 0.$$

(i) By direct calculation, it is obvious that

$$\begin{aligned} a_1(\xi) &= \mu[(|\gamma|-|\beta|)\xi_2^2 + k_s^2|\beta|] \\ &= \mu\frac{\xi_2^2(|\gamma|^2-|\beta|^2) + k_s^2(|\beta|^2+|\beta||\gamma|)}{|\gamma|+|\beta|} \\ &= \mu\frac{\xi_1^2k_s^2 + \xi_2^2k_p^2 + k_s^2|\beta||\gamma| - k_p^2k_s^2}{|\gamma|+|\beta|} \\ &\ge \mu\frac{(|\xi|^2-k_s^2)k_p^2 + k_s^2|\beta||\gamma|}{|\gamma|+|\beta|} > 0. \end{aligned} \tag{3.3}$$

Here the condition $|\xi| > K\omega > k_s$ is used in the last step.

(ii) Denote $g(\xi) = (|\gamma| - |\beta|)|\xi|^2 + k_s^2|\beta|$. Similarly to (3.3) we have $g(\xi) > 0$. Then one arrives at

$$\begin{aligned}&\begin{vmatrix} a_1(\xi) & b(\xi) \\ b(\xi) & a_2(\xi) \end{vmatrix}\\ &= a_1a_2 - b^2\\ &= \mu^2[(|\gamma| - |\beta|)|\xi_1|^2 + k_s^2|\beta|][(|\gamma| - |\beta|)|\xi_2|^2 + k_s^2|\beta|] - \mu^2\xi_1^2\xi_2^2(|\gamma| - |\beta|)^2\\ &= \mu^2k_s^2|\beta|g(\xi) > 0.\end{aligned}$$

(iii) Denote $h(\xi) = 2\xi_1\xi_2b(\xi) - a_1(\xi)\xi_2^2 - a_2(\xi)\xi_1^2$, then it can be verified that

$$\begin{aligned}\det(\mathcal{M}_1(\xi)) &= a_3(\xi)\begin{vmatrix} a_1(\xi) & b(\xi) \\ b(\xi) & a_2(\xi) \end{vmatrix} + (-a_1(\xi)c(\xi)^2\xi_2^2 + 2b(\xi)c(\xi)^2\xi_1\xi_2 - a_2(\xi)c(\xi)^2\xi_1^2)\\ &= \mu^2k_s^2|\beta||\gamma|g(\xi)\omega^2 + c(\xi)^2h(\xi). \qquad (3.4)\end{aligned}$$

Direct calculation implies

$$\begin{aligned}h(\xi) =& -2\mu\xi_1^2\xi_2^2(|\gamma| - |\beta|) - \mu\xi_1^2[\xi_1^2(|\gamma| - |\beta|) + k_s^2|\beta|]\\ &- \mu\xi_2^2[\xi_2^2(|\gamma| - |\beta|) + k_s^2|\beta|]\\ =& -\mu(|\gamma| - |\beta|)|\xi|^4 - \mu k_s^2|\beta||\xi|^2 = -\mu|\xi|^2g(\xi). \qquad (3.5)\end{aligned}$$

Combining (3.4)–(3.5) gives

$$\begin{aligned}\det(\mathcal{M}_1(\xi)) &= \mu^3g(\xi)\{k_s^4|\beta||\gamma| - |\xi|^2[2|\gamma|(|\gamma| - |\beta|) + k_s^2]^2\}\\ &= \mu^3g(\xi)\left\{k_s^4|\beta||\gamma| - \left[\frac{2|\gamma|(k_p^2 - k_s^2)}{|\beta| + |\gamma|} + k_s^2\right]^2\right\}\\ &= \mu^3g(\xi)\frac{d(\xi)}{(|\gamma| + |\beta|)^2}\end{aligned}$$

with

$$d(\xi) = k_s^4(|\gamma||\beta| - |\xi|^2)(|\gamma| + |\beta|)^2 + 4|\gamma|(k_s^2 - k_p^2)(|\gamma|k_p^2 + |\beta|k_s^2)|\xi|^2.$$

Hence we only need to verify $d(\xi) > 0$ for $|\xi| > K\omega$. Taking $|\xi|^2 = K'k_s^2$ implies

$$\begin{aligned}d(\xi) =& k_s^8[(\sqrt{(K' - \alpha)(K' - 1)} - K')(\sqrt{K' - 1} + \sqrt{K' - \alpha})^2\\ &+ 4(1 - \alpha)K'\sqrt{K' - 1}(\sqrt{K' - \alpha} + \alpha\sqrt{K' - 1})]\\ >& k_s^8[-(\sqrt{K' - \alpha} + \sqrt{K' - 1})^2 + 4(1 - \alpha)K'\sqrt{K' - 1}\alpha(\sqrt{K' - \alpha} + \sqrt{K' - 1})]\end{aligned}$$

with $\alpha := k_p^2/k_s^2 = \mu/(\lambda + 2\mu) < 1$. In order to show $d(\xi) > 0$, we will verify

$$2(1 - \alpha)\alpha K'\sqrt{K' - 1} > \sqrt{K' - \alpha},$$

i.e.,

$$K'\sqrt{\frac{K' - 1}{K' - \alpha}} > \frac{1}{2(1 - \alpha)\alpha}. \qquad (3.6)$$

For $|\xi| > K\omega = \omega(\lambda+2\mu)/(\mu\sqrt{\lambda+\mu})$, we can verify that $K' = |\xi|^2/k_s^2$ has the lower bound

$$K' > \max\left\{\frac{4-\alpha}{3}, \frac{1}{\alpha(1-\alpha)}\right\} = \frac{1}{\alpha(1-\alpha)} = \frac{(\lambda+2\mu)^2}{\mu(\lambda+\mu)}.$$

Then we have

$$\sqrt{\frac{K'-1}{K'-\alpha}} > \frac{1}{2}, \quad K' > \frac{1}{\alpha(1-\alpha)},$$

which guarantee (3.6). In summary, the assumption $|\xi| > K\omega$ leads to $d(\xi) > 0$, which gives $\det \mathcal{M}_1(\xi) > 0$.

(3) For $\rho(\xi) = |\xi|^2 + \beta\gamma$, direct calculation gives

$$\begin{cases} k_p^2 \le |\rho| \le k_p k_s, \quad 0 \le |\xi| \le k_p, \\ k_p^2 \le |\rho| \le k_s^2, \quad k_p \le |\xi| \le k_s, \\ c_K\omega^2 \le |\rho| \le k_s^2, \quad k_s \le |\xi| \le K\omega, \end{cases} \tag{3.7}$$

with

$$c_K = K^2 - \sqrt{(K^2-1/\mu)(K^2-1/(\lambda+2\mu))} > 1/(\lambda+2\mu).$$

Here to derive the inequality for $k_s \le |\xi| \le K\omega$ we have used the fact that the function

$$\rho(\xi) = |\xi|^2 - \sqrt{|\xi|^2-k_p^2}\sqrt{|\xi|^2-k_s^2}$$

is decreasing with respect to $|\xi|$ for $|\xi| \ge k_s$. We also compute $\gamma-\beta$ as

$$\gamma-\beta = \sqrt{k_s^2-|\xi|^2} - \sqrt{k_p^2-|\xi|^2} = \begin{cases} |\gamma|-|\beta|, \quad 0 < |\xi| \le k_p, \\ |\gamma| - i|\beta|, \quad k_p < |\xi| \le k_s, \\ i(|\gamma|-|\beta|), \quad |\xi| > k_s. \end{cases}$$

Then we immediately obtain

$$\begin{cases} |\gamma-\beta| \le \sqrt{k_s^2-k_p^2}, & 0 < |\xi| \le k_p \text{ or } |\xi| > k_s, \\ |\gamma-\beta| = \sqrt{|\gamma|^2+|\beta|^2} = \sqrt{k_s^2-k_p^2}, & k_p < |\xi| \le k_s. \end{cases} \tag{3.8}$$

To prove the third result, it suffices to verify the inequality $M_{ij} \le C\omega$ for $i,j = 1,2,3$ and $|\xi| \le K\omega$. For M_{33}, by (3.7) we have

$$|M_{33}| = \left|\frac{\gamma\omega^2}{\rho}\right| \le \begin{cases} \omega^2 k_s/k_p^2 = \omega(\lambda+2\mu)/\sqrt{\mu}, & 0 \le |\xi| \le k_p, \\ \omega^2\sqrt{k_s^2-k_p^2}/k_p^2 = \omega\sqrt{(\lambda+\mu)(\lambda+2\mu)/\mu}, & k_p \le |\xi| \le k_s, \\ \omega^2\sqrt{K^2\omega^2-k_s^2}/c_K\omega^2 = \omega\sqrt{K^2-1/\mu}/c_K, & k_s \le |\xi| \le K\omega. \end{cases} \tag{3.9}$$

Similarly, M_{23} and M_{32} can be estimated using (3.7) by

$$|M_{23}| = |M_{32}|$$

(3.10)
$$= \left|\frac{2\mu\rho\xi_2 - \omega^2\xi_2}{\rho}\right| \le \begin{cases} 2\mu k_p + \omega^2/k_p = \omega(2\mu/\sqrt{\lambda+2\mu} + \sqrt{\lambda+2\mu}), & 0 \le |\xi| \le k_p, \\ 2\mu k_s + \omega k_s/k_p^2 = \omega(2\sqrt{\mu} + (\lambda+2\mu)/\sqrt{\mu}), & k_p \le |\xi| \le k_s, \\ 2\mu K\omega + K\omega^3/c_K\omega^2 = \omega(2\mu K + K/c_K), & k_s \le |\xi| \le K\omega. \end{cases}$$

It is obvious that $|M_{13}| = |M_{31}|$ can also be estimated by the right-hand side of (3.10). It remains to estimate M_{11}, M_{22}, M_{12}, and M_{21}. For convenience, we write

$$\sqrt{k_s^2 - k_p^2} = C_{\lambda,\mu}\omega \quad \text{with} \quad C_{\lambda,\mu} := \sqrt{\frac{\lambda+\mu}{\mu(\lambda+2\mu)}}.$$

Combining (3.7)–(3.8) gives

$$|M_{11}| \le \mu\left|\frac{(\gamma-\beta)\xi_2^2 + k_s^2\beta}{\rho}\right|$$

(3.11)
$$\le \begin{cases} \mu(\omega C_{\lambda,\mu} + k_s^2/k_p) = \omega(\mu C_{\lambda,\mu} + \sqrt{\lambda+2\mu}), & 0 \le |\xi| \le k_p, \\ \mu(C_{\lambda,\mu}\omega k_s^2/k_p^2 + \omega C_{\lambda,\mu}k_s^2/k_p^2) = 2\omega C_{\lambda,\mu}(\lambda+2\mu), & k_p \le |\xi| \le k_s, \\ \omega(\mu C_{\lambda,\mu}K^2/c_K + \sqrt{K^2 - 1/(\lambda+2\mu)}/c_K), & k_s \le |\xi| \le K\omega. \end{cases}$$

Obviously, $|M_{22}|$ can also be estimated by the right-hand side of (3.11). For $|M_{12}|$ and $|M_{21}|$, we combine (3.7)–(3.8) to obtain

$$|M_{12}| = |M_{21}|$$

(3.12)
$$\le \mu\left|\frac{\xi_1\xi_2(\gamma-\beta)}{\rho}\right| \le \begin{cases} \omega\mu C_{\lambda,\mu}, & 0 \le |\xi| \le k_p, \\ \mu k_s^2\omega C_{\lambda,\mu}/k_p^2 = \omega(\lambda+2\mu)C_{\lambda,\mu}, & k_p \le |\xi| \le k_s, \\ \mu K^2 C_{\lambda,\mu}\omega^3/c_K\omega^2 = \omega(\mu K^2 C_{\lambda,\mu})/c_K, & k_s \le |\xi| \le K\omega. \end{cases}$$

Combining the above results (3.9)–(3.12), we have

$$\|M\| \le \begin{cases} C_{K,1}\omega, & 0 \le |\xi| \le k_p, \\ C_{K,2}\omega, & k_p \le |\xi| \le k_s, \\ C_{K,3}\omega, & k_s \le |\xi| \le K\omega, \end{cases}$$

with

$$C_{K,1} = \max\left\{\frac{\lambda+2\mu}{\sqrt{\mu}}, \frac{2\mu}{\sqrt{\lambda+2\mu}} + \sqrt{\lambda+2\mu}, \mu C_{\lambda,\mu} + \sqrt{\lambda+2\mu}, \mu C_{\lambda,\mu}\right\},$$
$$C_{K,2} = \max\left\{\sqrt{\frac{(\lambda+\mu)(\lambda+2\mu)}{\mu}}, (\lambda+2\mu)C_{\lambda,\mu}, 2\sqrt{\mu} + \frac{\lambda+2\mu}{\sqrt{\mu}}, 2C_{\lambda,\mu}(\lambda+2\mu)\right\},$$
$$C_{K,3} = \max\left\{\frac{\sqrt{K^2 - \frac{1}{\mu}}}{c_K}, 2\mu K + \frac{K}{c_K}, \frac{\mu K^2 C_{\lambda,\mu}}{c_K}, \frac{\mu K^2 C_{\lambda,\mu}}{c_K} + \frac{\sqrt{K^2 - \frac{1}{\lambda+2\mu}}}{c_K}\right\}.$$

It can be verified that

$$C_{K,1} \le 2\frac{\lambda+2\mu}{\sqrt{\mu}}, \quad C_{K,2} \le \frac{\lambda+2\mu}{\sqrt{\mu}} + 2(\lambda+2\mu)C_{\lambda,\mu} + 2\sqrt{\mu},$$

and

$$C_{K,3} \le \frac{K}{c_K} + \frac{\mu K^2 C_{\lambda,\mu}}{c_K} + 2\mu K.$$

Recalling that $c_K > 1/(\lambda+2\mu)$, we have

$$\begin{aligned}\max\{C_{K,1}, C_{K,2}, C_{K,3}\} &\leq 2(\lambda+4\mu)K + (\mu(\lambda+2\mu)K^2 + 2(\lambda+2\mu))C_{\lambda,\mu}\\ &= 2(\lambda+4\mu)K + (\mu(\lambda+2\mu)K^2 + 2(\lambda+2\mu))\sqrt{\frac{\lambda+\mu}{\mu(\lambda+2\mu)}}.\end{aligned}$$

The proof is completed. ∎

Recall that there exists a constant $C_0 = C_0(h, L, m, M) > 0$ independent of ω such that

$$\|\nabla u\|^2_{L^2(S_h)^3} \geq 1/C_0\, ||u||^2_{V_h}, \qquad ||u||^2_{H^{1/2}(\Gamma_h)} \leq C_0\, ||u||^2_{V_h} \tag{3.13}$$

for all $u \in V_h$. The well-posedness result for small frequencies is stated below.

THEOREM 3.3. *Let $K, C_K > 0$ be given as in Lemma* 3.1. *Then there exists a small frequency $\omega_0 > 0$ such that the variational problem admits a unique solution in V_h for all $\omega \in (0, \omega_0]$.*

Proof. It is clear that $\|\nabla\times u\|^2_{L^2(S_h)^3} \leq \|\nabla u\|^2_{L^2(S_h)^3}$. Now it follows from the definition of B and Lemma 3.1 that

$$\begin{aligned}\Re B(u,u) &= 2\mu\|\nabla u\|^2_{L^2(S_h)^3} + \lambda\|\nabla\cdot u\|^2_{L^2(S_h)} - \mu\|\nabla\times u\|^2_{L^2(S_h)^3}\\ &\quad - \omega^2\|u\|^2_{L^2(S_h)^3} - \Re\int_{\Gamma_h}\bar{u}\cdot\mathcal{T}u\mathrm{d}s\\ &= 2\mu\|\nabla u\|^2_{L^2(S_h)^3} + \lambda\|\nabla\cdot u\|^2_{L^2(S_h)} - \mu\|\nabla\times u\|^2_{L^2(S_h)^3} - \omega^2\|u\|^2_{L^2(S_h)^3} \qquad (3.14)\\ &\quad + \int_{|\xi|\leq K\omega}\mathrm{Re}(-\mathrm{i}\mathcal{M}(\xi))\hat{u}\cdot\bar{\hat{u}}\mathrm{d}\xi + \int_{|\xi|>K\omega}\mathrm{Re}(-\mathrm{i}\mathcal{M}(\xi))\hat{u}\cdot\bar{\hat{u}}\mathrm{d}\xi\\ &\geq \mu\|\nabla u\|^2_{L^2(S_h)^3} - \omega^2\|u\|^2_{L^2(S_h)^3} + \int_{|\xi|\leq K\omega}\mathrm{Re}(-\,\mathrm{i}\mathcal{M}(\xi))\hat{u}\cdot\bar{\hat{u}}\mathrm{d}\xi\\ &\geq \mu\|\nabla u\|^2_{L^2(S_h)^3} - \omega^2\|u\|^2_{L^2(S_h)^3} - C_K\, C_0\omega\|u\|^2_{V_h}, \qquad (3.15)\end{aligned}$$

where the constant $C_K > 0$ is given by Lemma 3.1.3 and the constant C_0 is specified in (3.13). By Lemma 3.4 in [7] we have the following Poincare's inequality,

$$\|u\|^2_{L^2(S_h)^3} \leq (h-m)\|\partial_3 u\|^2_{L^2(S_h)^3} \leq (h-m)\|\nabla u\|^2_{L^2(S_h)^3}, \quad u \in V_h. \tag{3.16}$$

Using (3.14)–(3.16), we obtain the estimate

$$\begin{aligned}\Re B(u,u) &\geq \left(\mu/C_0 - \omega C_0\, C_K - \omega^2(h-m)\right)\|u\|^2_{V_h}\\ &\geq \left(\mu/C_0 - \omega_0 C_0\, C_K - \omega_0^2(h-m)\right)\|u\|^2_{V_h}\end{aligned}$$

for all $u \in V_h$ and $\omega \in (0, \omega_0]$. Choose ω_0 sufficiently small such that

$$\mu/C_0 - \omega_0 C_0\, C_K - \omega_0^2(h-m) > 0.$$

The proof is completed by applying the Lax–Milgram theorem. ∎

4. An a priori bound for smooth surfaces. In this section, we establish an a priori bound for a smooth surface at any frequency. The attractive feature is that all constants in the a priori estimates are bounded by explicit functions of ω, h, m, M, and L.

LEMMA 4.1. *Let $u \in V_h$ be a variational solution to (2.17) with $g \in V_h$. We have*

$$\|\nabla \cdot u\|^2_{L^2(\Gamma)},\ \|\nabla \times u\|^2_{L^2(\Gamma)^3} \le C_1\, \|g\|_{L^2(S_h)^3}\|\partial_3 u\|_{L^2(S_h)^3},$$

where $C_1 = 4\mu^{-1}(1+L^2)^{1/2}(\omega/\sqrt{\mu}\,(h-m)+1)$.

Proof. By [13, Lemma 4.1], we have the following Rellich identity:

$$\begin{aligned} &2\Re\int_{S_h}(\mu\Delta u + (\lambda+\mu)\nabla\nabla\cdot u + \omega^2 u)\cdot\partial_3\bar{u}\mathrm{d}x \\ &\quad = \left(-\int_{\Gamma} + \int_{\Gamma_h}\right)\left\{2\Re(Tu\cdot\partial_3\bar{u}) - \nu_3\mathcal{E}(u,\bar{u}) + \omega^2|u|^2\right\}\mathrm{d}s. \end{aligned} \tag{4.1}$$

Noticing that $u|_\Gamma = 0$ implies $\nu\times\nabla u_j|_\Gamma = 0$ for $j=1,2,3$, by direct calculations (see [11, Lemma 5]) we have

$$Tu\cdot\partial_3\bar{u} = \nu_3\mathcal{E}(u,\bar{u}) = \mu|\partial_\nu u|^2\nu_3 + \nu_3(\lambda+\mu)|\nabla\cdot u|^2 \quad \text{on} \quad \Gamma. \tag{4.2}$$

From [13, Lemma 4.2(ii)] we also have the following two identities:

$$\begin{aligned} &\int_{\Gamma_h}\left\{2\Re(Tu\cdot\partial_3\bar{u}) - \mathcal{E}(u,\bar{u}) + \omega^2|u|^2\right\}\mathrm{d}s \\ &\quad = 2\omega^2\int_{|\xi|<k_\mathrm{p}}\beta^2(\xi)|A_\mathrm{p}(\xi)|^2\,\mathrm{d}\xi + 2\mu\int_{|\xi|<k_\mathrm{s}}\gamma^2(\xi)|\mathbf{A}_\mathrm{s}(\xi)|^2\,\mathrm{d}\xi \\ &\quad = 2\omega^2\left\{\int_{|\xi|<k_\mathrm{p}}\beta^2(\xi)|A_\mathrm{p}(\xi)|^2\,\mathrm{d}\xi + \int_{|\xi|<k_\mathrm{s}}\gamma^2(\xi)|\tilde{\mathbf{A}}_\mathrm{s}(\xi)|^2\,\mathrm{d}\xi\right\}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \Im\int_{\Gamma_h}Tu\cdot\bar{u}\mathrm{d}s &= \int_{|\xi|<k_\mathrm{p}}\omega^2\beta(\xi)|A_\mathrm{p}(\xi)|^2\mathrm{d}\xi + \int_{|\xi|<k_\mathrm{s}}\mu\gamma(\xi)|\boldsymbol{A}_\mathrm{s}(\xi)|^2\mathrm{d}\xi \\ &= \omega^2\left\{\int_{|\xi|<k_\mathrm{p}}\beta(\xi)|A_\mathrm{p}(\xi)|^2\mathrm{d}\xi + \int_{|\xi|<k_\mathrm{s}}\gamma(\xi)|\tilde{\boldsymbol{A}}_\mathrm{s}(\xi)|^2\mathrm{d}\xi\right\}. \end{aligned} \tag{4.4}$$

Here we have used the relation $|\boldsymbol{A}_\mathrm{s}(\xi)|^2 = k_\mathrm{s}^2|\tilde{\boldsymbol{A}}_\mathrm{s}(\xi)|^2$. Note that the identity (4.3) corrects a mistake made in [13, formula (4.1)]. Hence, combining (4.1) and (4.2) gives

$$\begin{aligned} &-\int_\Gamma \mu|\partial_\nu u|^2\nu_3 + \nu_3(\lambda+\mu)|\nabla\cdot u|^2\,\mathrm{d}s \\ &\quad = \int_{\Gamma_h}2\Re(Tu\cdot\partial_3\bar{u}) - \nu_3\mathcal{E}(u,\bar{u}) + \omega^2|u|^2\,\mathrm{d}s - 2\Re\int_{S_h}g\partial_3\bar{u}\,\mathrm{d}x. \end{aligned} \tag{4.5}$$

Using (4.3) and (4.4) and taking the imaginary part of (2.17), we get

$$\begin{aligned} \int_{\Gamma_h}\left\{2\Re(Tu\cdot\partial_3\bar{u}) - \mathcal{E}(u,\bar{u}) + \omega^2|u|^2\right\}\mathrm{d}s &\le 2k_\mathrm{s}\Im\int_{\Gamma_h}Tu\cdot\bar{u}\mathrm{d}s \\ &\le 2k_\mathrm{s}\Im\int_{S_h}g\cdot\bar{u}\mathrm{d}s. \end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6) then gives the estimates

$$
\begin{aligned}
&-\int_\Gamma \mu|\partial_\nu u|^2\nu_3 + \nu_3(\lambda+\mu)|\nabla\cdot u|^2\,\mathrm{d}s \\
&\le 2k_s\Im\int_{S_h} g\cdot\bar u\mathrm{d}x - 2\Re\int_{S_h} g\cdot\partial_3\bar u\mathrm{d}x \\
&\le 2\|g\|^2_{L^2(S_h)^3}\left(\frac{\omega}{\sqrt{\mu}}\|u\|^2_{L^2(S_h)^3} + \|\partial_3 u\|^2_{L^2(S_h)^3}\right) \\
&\le 2(\omega/\sqrt{\mu}\,(h-m)+1)\|g\|^2_{L^2(S_h)^3}\|\partial_3 u\|^2_{L^2(S_h)^3}, \qquad (4.7)
\end{aligned}
$$

where the last identity follows from (3.16). Since

$$
\nu_3(x) = -\frac{1}{\sqrt{1+|\nabla_{x'}f|^2}} < -(1+L^2)^{-1/2} < 0 \quad \text{on } \Gamma, \qquad (4.8)
$$

from (4.7) we obtain that

$$
\begin{aligned}
&\|\nabla\cdot u\|^2_{L^2(\Gamma)} + \|\partial_\nu u\|^2_{L^2(\Gamma)^3} \\
&\le 2\mu^{-1}(1+L^2)^{1/2}(\omega/\sqrt{\mu}\,(h-m)+1)\|g\|^2_{L^2(S_h)^3}\|\partial_3 u\|^2_{L^2(S_h)^3}. \qquad (4.9)
\end{aligned}
$$

Finally, using $u=0$ on Γ and the identities in [11, (4.17)] we have

$$
\nu_3|\nabla\times u|^2 = \nu_3(|\nabla u|^2 - |\nabla\cdot u|^2) = \nu_3(|\partial_\nu u|^2 - |\nabla\cdot u|^2) \quad \text{on } \Gamma.
$$

Thus, $\|\nabla\times u\|_{L^2(\Gamma)^3}$ can also be bounded by the right-hand side of (4.9) multiplied by two. ∎

We next need to derive estimates for the L^2-norms of the scalar function $\nabla\cdot u$ and the vector function $\nabla\times u$ on the artificial boundary Γ_H and also over the strip S_H with $H=h+1$. The derivation is based on the a priori bound for the Helmholtz equation in [7]. By (2.5), we define

$$
\varphi := -\frac{\mathrm{i}}{k_p^2}\nabla\cdot u, \quad \psi := \frac{\mathrm{i}}{k_s^2}\nabla\times u \qquad \text{in} \quad x_3 > h.
$$

Since both φ and ψ satisfy the Helmholtz equation (2.6) and the UASR (2.8), one has the following DtN map on the artificial boundary Γ_H,

$$
\widetilde{\mathcal{T}}w = \mathcal{F}^{-1}(\mathrm{i}\eta\mathcal{F}w), \quad w\in H^{1/2}(\Gamma_H)^d, \qquad (4.10)
$$

where $d=1$, $\eta=\beta$ if $w=\varphi$ and $d=3$, $\eta=\gamma$ if $w=\psi$. Moreover, $\widetilde{\mathcal{T}}$ is a bounded linear map of $H^{1/2}(\Gamma_H)^d$ to $H^{-1/2}(\Gamma_H)^d$ by [7, Lemma 2.4]. From Lemma 4.1 we can estimate the L^2-norm of the trace $w=\nabla\cdot u, \nabla\times u$ on Γ as

$$
\|w\|^2_{L^2(\Gamma)^d} \le C_1(\omega,h,L)\|g\|_{L^2(S_H)^3}\|\partial_3 u\|_{L^2(S_H)^3}. \qquad (4.11)
$$

The following lemma provides estimates for w on S_H and the trace of w on Γ_H.

Lemma 4.2. *Assume that w satisfies the Helmholtz equation*

$$
\Delta w + k^2 w = g_0 \quad \text{in } S_H, \qquad \partial_3 w = \widetilde{\mathcal{T}}w = \mathcal{F}^{-1}(i\sqrt{k^2-\xi^2}\mathcal{F}w) \quad \text{on } \Gamma_H, \qquad (4.12)
$$

where $g_0\in L^2(S_H)$. Then there holds the estimate

$$
\|w\|_{L^2(\Gamma_H)} \le \|w\|_{L^2(S_H)} \le \widetilde{C}_2(L,k,h)\|w\|_{L^2(\Gamma)} + \widetilde{C}_3(k,h)\|g_0\|_{L^2(S_H)} \qquad (4.13)
$$

with

$$\widetilde{C}_2(L,k,h) = C(1+L^2)^{1/4}\sqrt{H-m}(1+k(H-m))$$

and

$$\widetilde{C}_3(k,h) = C(H-m)(1+k(H-m))^2/k,$$

where the constant C is independent of $L, K, m,$ and h.

Proof. Consider the boundary value problem of finding $v \in H^1(S_H)$ such that

$$(\triangle + k^2)v = \bar{w} \quad \text{in} \quad S_H, \quad v = 0 \quad \text{on} \quad \Gamma, \quad \partial_3 v = \widetilde{T}v \quad \text{on} \quad \Gamma_H. \tag{4.14}$$

By [7, Lemma 4.6] the boundary value problem (4.14) is well-posed with the following estimate:

$$\|\nabla v\|_{L^2(S_H)} + k\|v\|_{L^2(S_H)} \le C(1+k(H-m))^2(H-m)\|w\|_{L^2(S_H)}. \tag{4.15}$$

We first prove that $\|\partial_\nu v\|^2_{L^2(\Gamma)^3} \le C\|w\|^2_{L^2(S_H)}$ for some constant $C > 0$ depending explicitly on $\omega, H,$ and the Lipschitz constant L of Γ. The Rellich identity for the Helmholtz equation gives

$$\begin{aligned} &2\Re \int_{S_H} \partial_3 \bar{v}(\Delta v + k^2 v)\mathrm{d}x \\ &\quad = \left(\int_\Gamma + \int_{\Gamma_H}\right)\{2\Re(\partial_\nu v \partial_3 \bar{v}) - \nu_3|\nabla v|^2 + \nu_3 k^2|v|^2\}\mathrm{d}s, \end{aligned} \tag{4.16}$$

which can be proved in the same way as (4.1). From [7, (4.13)] it holds that

$$\begin{aligned} &\int_{\Gamma_H}\{2\Re(\partial_\nu v \partial_3 \bar{v} - \nu_3|\nabla v|^2 + \nu_3 k^2|v|^2)\}\mathrm{d}s \le 2k\Im\int_{\Gamma_H}\bar{v}\widetilde{T}v\mathrm{d}s \\ &\quad \le 2k\Im\int_{S_H}\bar{v}\bar{w}\mathrm{d}x. \end{aligned} \tag{4.17}$$

Moreover, using the identities in (4.17) of [11] on Γ and the bound for ν_3 in (4.8) one has

$$\begin{aligned} &-\int_{\Gamma_H}\{2\Re(\partial_\nu v \partial_3 \bar{v} - \nu_3|\nabla v|^2 + \nu_3 k^2|v|^2)\}\mathrm{d}s = -\int_\Gamma \nu_3|\partial_\nu v|^2\mathrm{d}s \\ &\quad \ge (1+L^2)^{-1/2}\|\partial_\nu v\|^2_{L^2(\Gamma)}. \end{aligned} \tag{4.18}$$

Plugging (4.17) and (4.18) into (4.16) and using (4.15) yield the estimate

$$\begin{aligned} \|\partial_\nu v\|^2_{L^2(\Gamma)} &\le (1+L^2)^{1/2}\left\{-2\Re\int_{S_H}\bar{w}\partial_3 v\mathrm{d}x + 2k\Im\int_{S_H}\bar{w}\bar{v}\mathrm{d}x\right\} \\ &\le 2(1+L^2)^{1/2}\|w\|_{L^2(S_H)}(k\|v\|_{L^2(S_H)} + \|\nabla v\|_{L^2(S_H)}) \\ &\le C(1+L^2)^{1/2}(H-m)(1+k(H-m))^2\|w\|^2_{L^2(S_H)}, \end{aligned} \tag{4.19}$$

where the constant C is independent of w and L, m, h, k.

Now we prove the second inequality in (4.13). Following the approach of [10, Lemma 7], we obtain that

$$\begin{aligned}\int_{S_H}\{w\Delta v - v\Delta w\}\mathrm{d}x &= \int_{\Gamma_H}\{w\partial_\nu v - v\partial_\nu w\}\mathrm{d}s + \int_\Gamma w\partial_\nu \mathrm{d}s\\ &= \int_{\Gamma_H}\{w\widetilde{T}v - v\widetilde{T}w\}\mathrm{d}s + \int_\Gamma w\partial_\nu v\mathrm{d}s\\ &= \int_\Gamma w\partial_\nu v\mathrm{d}s.\end{aligned}$$

Note that $v=0$ on Γ, and the DtN operator $\widetilde{T}$ defined in (4.10) is symmetric (see Lemma 3.2 in [7]). Thus,

$$\begin{aligned}\int_{S_H}|w|^2\mathrm{d}x &= \int_{S_H} w(\Delta v + k^2 v)\mathrm{d}x\\ &= \int_{S_H} v(\Delta w + k^2 w)\mathrm{d}x + \int_\Gamma w\partial_\nu v\mathrm{d}s\\ &= \int_{S_H} v g_0\mathrm{d}x + \int_\Gamma w\partial_\nu v\mathrm{d}s.\end{aligned}$$

Noting (4.15) and (4.19) one has

$$\begin{aligned}\|w\|^2_{L^2(S_H)} \le& \|v\|^2_{L^2(S_H)}\|g_0\|^2_{L^2(S_H)} + \|w\|^2_{L^2(\Gamma)}\|\partial_\nu v\|^2_{L^2(\Gamma)}\\ \le& C\sqrt{H-m}(1+L^2)^{1/4}(1+k(H-m))\|w\|_{L^2(S_H)}\|w\|_{L^2(\Gamma)}\\ &+ C(H-m)\frac{(1+k(H-m))^2}{k}\|w\|_{L^2(S_H)}\|g_0\|_{L^2(S_H)}.\end{aligned}$$

Then the following inequality is proved:

$$\|w\|_{L^2(S_H)} \le \widetilde{C}_2(L,k,h)\|w\|_{L^2(\Gamma)} + \widetilde{C}_3(k,h)\|g_0\|_{L^2(S_H)}. \tag{4.20}$$

To estimate the first inequality in (4.13) we use

$$\int_{\Gamma_H}|w|^2\mathrm{d}s \le \int_{\Gamma_c}|w|^2\mathrm{d}s \quad \text{for all } c\in(h,H],$$

which follows from the proof of [7, Lemma 2.2]. Then we have

$$(H-h)\int_{\Gamma_H}|w|^2\mathrm{d}x \le \int_{S_H\setminus S_h}|w|^2\mathrm{d}s \le \int_{S_H}|w|^2\mathrm{d}s. \tag{4.21}$$

The estimate (4.13) is proved by combing (4.20) and (4.21). ∎

Next we prove the estimates of the L^2 norms of $\nabla\cdot u$ and $\nabla\times u$ on S_H and Γ_H. Using Lemma 4.2 for $w=\nabla\cdot u$ and $\nabla\times u$ with $g_0=(\lambda+2\mu)\nabla\cdot g$ and $\mu\nabla\times g$ in (4.12), respectively, and (4.11), we obtain the estimate

$$\begin{aligned}&\|\nabla\cdot u\|^2_{L^2(S_H)} + \|\nabla\times u\|^2_{L^2(S_H)^3}\\ &\quad\le C_2(\omega,h,L)^2 C_1(\omega,h,L)\|g\|_{V_h}\|\partial_3 u\|_{L^2(S_H)^3} + C_3(\omega,h)^2\|g\|^2_{V_h},\end{aligned} \tag{4.22}$$

where

$$C_2(\omega,h,L) = C(1+L^2)^{1/4}\sqrt{H-m}(1+\omega(H-m))$$

and

$$C_3(\omega,h) = C(H-m)(1+\omega(H-m))^2/\omega.$$

In a similar way, from the estimates (4.13) and (4.11) we have the bound

$$\begin{aligned} &\|\nabla\cdot u\|^2_{L^2(\Gamma_H)} + \|\nabla\times u\|^2_{L^2(\Gamma_H)^3} \\ &\quad \le C_2(\omega,h,L)^2 C_1(\omega,h,L)\|g\|_{V_h}\|\partial_3 u\|_{L^2(S_H)^3} + C_3(\omega,h)^2\|g\|^2_{V_h}. \end{aligned} \tag{4.23}$$

The following theorem provides the a priori bound for the solution to *variational problem* I dependent on the frequency and geometry of the rough surface.

THEOREM 4.3. *Assume that* Γ *is given by the graph of a Lipschitz function* f *satisfying* (2.2), *and that* $u\in V_h$ *is a solution to the variational problem* (2.17). *Then there exist a constant* C *independent of* $\omega, h,$ *and the Lipschitz constant* L *of* f *such that the following a priori bound holds,*

$$\|u\|_{V_h} \le (h-m+2)(C_4(\omega,h)+C_5(\omega,h)+C_6(\omega,h,L))\|g\|_{V_h},$$

where

$$\begin{aligned} &C_4(\omega,h) = C(h+1-m)\omega, \quad C_5(\omega,h) = C\sqrt{1+\omega^{-1}}C_3(\omega,h), \\ &C_6(\omega,h) = C(\omega^{-1}+1)C_1(\omega,h,L)C_2(\omega,h,L)^2. \end{aligned}$$

Proof. We first assume that f is smooth. Multiplying both sides of the Navier equation by $(x_3-m)\partial_3\bar{u}$, integrating over S_H, and using integration by parts yield

$$\begin{aligned} &2\Re\int_{S_H}(\triangle^*+\omega^2)u\cdot(x_3-m)\partial_3\bar{u}\mathrm{d}x \\ &\quad = \int_{S_H}\left\{\mathcal{E}(u,\bar{u}) - 2\Re\left\{\sum_{j=1}^3 \mathcal{E}(u,(x_3-m)e_j)\partial_3\bar{u}_j\right\} - \omega^2|u|^2\right\}\mathrm{d}x \\ &\qquad + \left(\int_{\Gamma_H} + \int_\Gamma\right)[-\nu_3\mathcal{E}(u,\bar{u}) + 2\Re(Tu\cdot\partial_3\bar{u}) + \nu_3\omega^2|u|^2](x_3-m)\mathrm{d}s. \end{aligned} \tag{4.24}$$

Letting $v=u$ in the variational formulation (2.17) gives

$$\begin{aligned} &\int_{S_H}\{\mathcal{E}(u,\bar{u}) - \omega^2|u|\}\mathrm{d}x - \Re\int_{|\xi|>K\omega} \mathrm{i}\mathcal{M}(\xi)\hat{u}(\xi,H)\cdot\bar{\hat{u}}(\xi,H)\mathrm{d}\xi \\ &\quad = -\Re\int_{S_H} g\cdot\bar{u}\mathrm{d}x + \Re\int_{|\xi|\le K\omega} \mathrm{i}\mathcal{M}(\xi)\hat{u}(\xi,H)\cdot\bar{\hat{u}}(\xi,H)\mathrm{d}\xi. \end{aligned} \tag{4.25}$$

Taking the real part and using Lemma 3.1 we have

$$\begin{aligned} &\int_{S_H}\{\mathcal{E}(u,\bar{u}) - \omega^2|u|\}\mathrm{d}x \\ &\quad \le -\Re\int_{S_H} g\cdot\bar{u}\mathrm{d}x + \Re\int_{|\xi|\le K\omega} \mathrm{i}\mathcal{M}(\xi)\hat{u}(\xi,H)\cdot\bar{\hat{u}}(\xi,H)\mathrm{d}\xi. \end{aligned} \tag{4.26}$$

From (4.24) and using (4.26) and (4.2), we have

$$\begin{aligned}
&\int_{S_H} 2\Re\left\{\sum_{j=1}^{3}\mathcal{E}(u,(x_3-m)e_j)\partial_3\bar{u}_j\right\}\mathrm{d}x \\
&\qquad -\int_{\Gamma}(x_3-m)\{\mu|\partial_\nu u|^2+(\lambda+\mu)|\nabla\cdot u|^2\}\nu_3\mathrm{d}s \\
&\quad =\int_{S_H}\{\mathcal{E}(u,\bar{u})-\omega^2|u|^2\}\mathrm{d}x-2\Re\int_{S_H}(\triangle^*+\omega^2)u\cdot(x_3-m)\partial_3\bar{u}\mathrm{d}x \\
&\qquad +(H-m)\int_{\Gamma_H}\{2\Re(Tu\cdot\partial_3\bar{u})-\mathcal{E}(u,\bar{u})+\omega^2|u|^2\}\mathrm{d}s \\
&\quad \le\int_{S_H}\{-\Re g\cdot\overline{u}-2\Re(g\cdot\partial_3\bar{u})(x_3-m)\}\mathrm{d}x+\Re\int_{|\xi|\le K\omega}\mathrm{i}\mathcal{M}(\xi)\,\hat{u}_H(\xi)\cdot\bar{\hat{u}}_H(\xi)\mathrm{d}\xi \\
&\qquad +(H-m)\int_{\Gamma_H}\{2\Re(Tu\cdot\partial_3\bar{u})-\mathcal{E}(u,\bar{u})+\omega^2|u|^2\}\mathrm{d}s.
\end{aligned}\tag{4.27}$$

As $\|\mathcal{M}(\xi)\|\le C\omega$ for all $|\xi|<K\omega$, one has

$$\Re\int_{|\xi|\le K\omega}\mathrm{i}\mathcal{M}(\xi)\,\hat{u}_H(\xi)\cdot\bar{\hat{u}}_H(\xi)\mathrm{d}\xi\le C\omega\int_{|\xi|\le K\omega}|\hat{u}_H(\xi)|^2\mathrm{d}\xi.\tag{4.28}$$

Using (4.28), the Plancherel identity and (4.23) give

$$\begin{aligned}
\Re\int_{|\xi|\le K\omega}\mathrm{i}\mathcal{M}(\xi)\,\hat{u}_H(\xi)\cdot\bar{\hat{u}}_H(\xi)\mathrm{d}\xi &\le C\omega k_s^2\int_{|\xi|\le K\omega}\{|A_{\mathrm{p}}(\xi)|^2+|\boldsymbol{A}_{\mathrm{s}}(\xi)|^2\}\mathrm{d}\xi \\
&\le C\omega k_s^2(\|A_p\|^2_{L^2(\mathbb{R}^2)}+\|\boldsymbol{A}_s\|^2_{L^2(\mathbb{R}^2)^3}) \\
&\le C\omega k_s^2(\|\varphi\|^2_{L^2(\mathbb{R}^2)}+\|\psi\|^2_{L^2(\mathbb{R}^2)^3}) \\
&\le C\omega^{-1}\|\nabla\cdot u\|^2_{L^2(\Gamma_H)}+\|\nabla\times u\|^2_{L^2(\Gamma_H)^3} \\
&\le C\omega^{-1}\left(C_2(\omega,h,L)^2C_1(\omega,h,L)\|g\|_{V_h}\|\partial_3 u\|_{L^2(S_H)^3}+C_3(\omega,h)^2\|g\|^2_{V_h}\right),
\end{aligned}\tag{4.29}$$

where φ and ψ are introduced by (2.5) which satisfy the Helmholtz equations (2.6).

From the estimates (4.6) and (3.16) we have the following estimate for the last term in (4.27):

$$\begin{aligned}
\int_{\Gamma_H}\{2\Re(Tu\cdot\partial_3\bar{u}-\mathcal{E}(u,\bar{u})+\omega^2|u|^2)\}\mathrm{d}s &\le 2k_{\,\mathrm{s}}\Im\int_{S_H}g\cdot\bar{u}\mathrm{d}x \\
&\le 2k_{\mathrm{s}}\|g\|_{V_h}\|u\|_{L^2(S_H)^3} \\
&\le C(H-m)k_{\mathrm{s}}\|g\|_{V_h}\|\partial_3 u\|_{L^2(S_H)^3}.
\end{aligned}\tag{4.30}$$

Combining (4.29)–(4.30) and (4.27) and noting that the second term in (4.27) is nonnegative, we have

$$\begin{aligned}
&\int_{S_H} 2\Re\left\{\sum_{j=1}^{3}\mathcal{E}(u,(x_3-m)e_j)\partial_3\bar{u}_j\right\}\mathrm{d}x \\
&\quad\le C(H-m)\omega\|g\|_{L^2(S_H)^3}\|\partial_3 u\|_{L^2(S_H)^3}+C(\omega^{-1}+1) \\
&\qquad\times\left(C_2(\omega,h,L)^2C_1(\omega,h,L)\|g\|_{V_h}\|\partial_3 u\|_{L^2(S_H)^3}+C_3(\omega,h)^2\|g\|^2_{V_h}\right),
\end{aligned}\tag{4.31}$$

where the Poincaré inequality (3.16) is used in the last step.

Direct calculations yield

$$\mathcal{E}(u,(x_3-m)e_1)\partial_3\bar{u}_1 = 2\mu|\partial_3 u_1|^2 - \mu(\partial_3 u_1 - \partial_1 u_3)\,\partial_3\overline{u}_1,$$
$$\mathcal{E}(u,(x_3-m)e_2)\partial_3\bar{u}_2 = 2\mu|\partial_3 u_2|^2 + \mu(\partial_2 u_3 - \partial_3 u_2)\,\partial_3\overline{u_2},$$
$$\mathcal{E}(u,(x_3-m)e_3)\partial_3\bar{u}_3 = (\lambda+2\mu)|\partial_3 u_3|^2 + \lambda(\partial_1 u_1 + \partial_2 u_2)\,\partial_3\overline{u}_3.$$

Hence,

$$\begin{aligned}
&\int_{S_H} 2\Re\left\{\sum_{j=1}^{3}\mathcal{E}(u,(x_3-m)e_j)\partial_3\bar{u}_j\right\}\mathrm{d}x \\
&\quad = 2(\lambda+2\mu)\|\partial_3 u_3\|^2_{L^2(S_H)^3} + 4\mu(\|\partial_3 u_1\|^2_{L^2(S_H)^3} + \|\partial_3 u_2\|^2_{L^2(S_H)^3}) \\
&\qquad + 2\lambda\left(\Re\int_{S_H}\partial_1 u_1\partial_3\bar{u}_3\mathrm{d}x + \Re\int_{S_H}\partial_2 u_2\partial_3\bar{u}_3\mathrm{d}x\right) \\
&\qquad - 2\mu\Re\left\{\int_{S_H}(\partial_3 u_1 - \partial_1 u_3)\partial_3\bar{u}_1 - (\partial_2 u_3 - \partial_3 u_2)\partial_3\bar{u}_2\mathrm{d}x\right\}.
\end{aligned} \tag{4.32}$$

Letting $C, \widetilde{C} > 0$ be two positive constants, we get

$$\begin{aligned}
&\int_{S_H} 2\Re\left\{\sum_{j=1}^{3}\mathcal{E}(u,(x_3-m)e_j)\partial_3\bar{u}_j\right\}\mathrm{d}x + C\|\nabla\cdot u\|^2_{L^2(S_H)} + \widetilde{C}\|\nabla\times u\|^2_{L^2(S_H)^3} \\
&\quad = I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{4.33}$$

where

$$\begin{aligned}
I_1 &:= [C+2(\lambda+2\mu)]\|\partial_3 u_3\|^2_{L^2(S_H)^3} + C\|\partial_1 u_1\|^2_{L^2(S_H)^3} + C\|\partial_2 u_2\|^2_{L^2(S_H)^3} \\
&\quad + (2C+2\lambda)\left(\Re\int_{S_H}\partial_1 u_1\partial_3\bar{u}_3\mathrm{d}x + \Re\int_{S_H}\partial_2 u_2\partial_3\bar{u}_3\mathrm{d}x\right) \\
&\quad + 2C(1-\epsilon)\Re\int_{S_H}\partial_1 u_1\partial_2\bar{u}_2\mathrm{d}x \\
&= \int_{S_H} A[\partial_1 u_1, \partial_2 u_2, \partial_3 u_3]^\top\cdot[\partial_1\bar{u}_1, \partial_2\bar{u}_2, \partial_3\bar{u}_3]^\top\mathrm{d}x, \\
A &:= \begin{pmatrix} C & C(1-\epsilon) & \lambda+C \\ C(1-\epsilon) & C & \lambda+C \\ \lambda+C & \lambda+C & C+2(\lambda+2\mu) \end{pmatrix}, \\
I_2 &:= 4\mu\|\partial_3 u_1\|^2_{L^2(S_H)^3} + \widetilde{C}\|\partial_3 u_1 - \partial_1 u_3\|^2_{L^2(S_H)} - 2\mu\,\Re\int_{S_H}(\partial_3 u_1 - \partial_1 u_3)\partial_3\bar{u}_1\mathrm{d}x, \\
I_3 &:= 4\mu\|\partial_3 u_2\|^2_{L^2(S_H)^3)} + \widetilde{C}\|\partial_2 u_3 - \partial_3 u_2\|^2_{L^2(S_H)} + 2\mu\,\Re\int_{S_H}(\partial_2 u_3 - \partial_3 u_2)\partial_3\bar{u}_2\mathrm{d}x, \\
I_4 &:= \widetilde{C}\|\partial_1 u_2 - \partial_2 u_1\|^2_{L^2(S_H)} + 2C\epsilon\Re\int_{S_H}\partial_1 u_1\partial_2\bar{u}_2\mathrm{d}x.
\end{aligned} \tag{4.34}$$

If $0 < \epsilon < 1$, then the matrix A is strictly positive provided $\mathrm{Det}(A) > 0$. A direct calculation gives

$$\mathrm{Det}(A) = C(-C^2\epsilon^2 + (8\mu\epsilon - 2(\lambda+2\mu)\epsilon^2)C - 2\epsilon\lambda^2).$$

Let ϵ be sufficiently small. Taking $C = \frac{4\mu}{\epsilon} - (\lambda + 2\mu) > 0$, we have

$$\mathrm{Det}(A) = C((4\mu - \epsilon(\lambda + 2\mu))^2 - 2\epsilon\lambda^2) > 0$$

if ϵ is chosen to be sufficiently small. Hence, the matrix $A \in \mathbb{R}^{3\times 3}$ is strictly positive for sufficiently small $\epsilon > 0$ and $C = \frac{4\mu}{\epsilon} - (\lambda + 2\mu)$. This gives

$$I_1 \geq C_0\,(\|\partial_1 u_1\|^2_{L^2(S_H)} + \|\partial_2 u_2\|^2_{L^2(S_H)} + \|\partial_3 u_3\|^2_{L^2(S_H)}), \tag{4.35}$$

where the constant $C_0 > 0$ only depends on λ and μ. By arguing in the same manner one has for $\widetilde{C} > \mu^2/4$ that

$$I_2 \geq C_0\,(\|\partial_3 u_1\|^2_{L^2(S_H)} + \|\partial_3 u_1 - \partial_1 u_3\|^2_{L^2(S_H)}), \tag{4.36}$$

$$I_3 \geq C_0\,(\|\partial_3 u_2\|^2_{L^2(S_H)} + \|\partial_3 u_2 - \partial_2 u_3\|^2_{L^2(S_H)}). \tag{4.37}$$

By using integration by parts we have

$$\begin{aligned}\int_{S_H} \partial_1 u_1 \partial_2 \bar{u}_2 \mathrm{d}x &= \int_{\Gamma_H} \nu_1 u_1 \partial_2 \bar{u}_2 \ \mathrm{d}s + \int_{\Gamma} \nu_1 u_1 \partial_2 \bar{u}_2 \mathrm{d}s - \int_{S_H} u_1 \partial_1 \partial_2 \bar{u}_2 \mathrm{d}x \\ &= -\int_{S_H} u_1 \partial_1 \partial_2 \bar{u}_2 \mathrm{d}x \\ &= -\int_{\Gamma_H} \nu_2 u_1 \partial_1 \bar{u}_2 \ \mathrm{d}s - \int_{\Gamma} \nu_2 u_1 \partial_1 \bar{u}_2 \mathrm{d}s + \int_{S_H} \partial_2 u_1 \partial_1 \bar{u}_2 \mathrm{d}x \\ &= \int_{S_H} \partial_2 u_1 \partial_1 \bar{u}_2 \mathrm{d}x,\end{aligned}$$

where we have used $u_1 = u_2 = 0$ on Γ and $\nu_1 = \nu_2 = 0$ on Γ_H. Then we have

$$\begin{aligned} I_4 &= \int_{S_H} B[\partial_1 u_2, \partial_2 u_1,]^\top \cdot [\partial_1 \bar{u}_2, \partial_2 \bar{u}_1,]^\top \mathrm{d}x, \\ B &:= \begin{pmatrix} \widetilde{C} & -\widetilde{C} + \epsilon C \\ -\widetilde{C} + \epsilon C & \widetilde{C} \end{pmatrix}. \end{aligned}$$

It is easy to see that B is strictly positive, which gives

$$I_4 \geq C_0(\|\partial_1 u_2\|^2_{L^2(S_H)} + \|\partial_2 u_1\|^2_{L^2(S_H)}) \tag{4.38}$$

if $\tilde{C} > \epsilon C = 4\mu - (\lambda + 2\mu)\epsilon$. Hence, it follows from (4.33)–(4.38) that there exist $C, \widetilde{C} > 0$ such that

$$\begin{aligned} &\int_{S_H} 2\Re \left\{ \sum_{j=1}^{3} \mathcal{E}(u, (x_3 - m)e_j) \partial_3 \bar{u}_j \right\} \mathrm{d}x + C\|\nabla \cdot u\|^2_{L^2(S_H)} + \widetilde{C}\|\nabla \times u\|^2_{L^2(S_H)^3} \\ &\quad \geq C_0\,\|\nabla u\|^2_{L^2(S_H)^3}. \end{aligned} \tag{4.39}$$

Using (4.22), (4.31), (4.39), and the Poincaré inequality (3.16) for u we obtain

$$||u||^2_{V_h} \leq (C_4(\omega, h)^2 + C_5(\omega, h)^2 + C_6(\omega, h, L)^2)\|g\|^2_{V_h}.$$

Now the a priori bound for a smooth profile has been proved. It can be extended to the case of a Lipschitz function by the method of approximation in [10]. Since the approximation arguments are standard, we omit the details for brevity. This completes the proof. ∎

5. Well-posedness for random rough surfaces. In this section, we investigate the well-posedness of elastic scattering by a random rough surface. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space. Denote by $S(\eta)$ a random surface

$$\Gamma(\eta) := \{x \in \mathbb{R}^3 : x_3 = f(\eta; x'), \eta \in \Omega, x' \in \mathbb{R}^2\}.$$

Similarly, $D(\eta)$ and $S_h(\eta)$ represent the random counterparts of D and S_h, respectively. Assume $f(\eta; x')$ is a Lipschitz continuous function with Lipschitz constant $L(\eta)$ for all $\eta \in \Omega$ and it also satisfies $m < f(\eta; x') < M$. The random source $g(\eta)$ is assumed to satisfy $g(\eta) \in L^2(D(\eta))^3$ with its support in $S_h(\eta)$. Similarly as in the deterministic case, we can give the following random boundary value problem:

$$\begin{aligned} \Delta^* u(\eta;\cdot) + \omega^2 u(\eta;\cdot) = g(\eta;\cdot) \quad &\text{in} \quad S_h(\eta), \\ u(\eta;\cdot) = 0 \quad &\text{on} \quad \Gamma(\eta), \\ Tu(\eta;\cdot) = \mathcal{T} u(\eta;\cdot) \quad &\text{on} \quad \Gamma_h. \end{aligned}$$

For simplicity, let $V_h(\eta) = V_h(S_h(\eta))$. Define a sesquilinear form $\tilde{B}_\eta$ on $V_h(\eta) \times V_h(\eta)$ by

$$\tilde{B}_\eta(u, v) = \int_{S_h(\eta)} \mathcal{E}(u, \bar{v}) - \omega^2 u \cdot \bar{v} \, \mathrm{d}x - \int_{\Gamma_h} \mathcal{T} u \cdot \bar{v} \, \mathrm{d}s, \tag{5.1}$$

and an antilinear functional $\tilde{G}_\eta$ on $V_h(\eta)$ by

$$\tilde{G}_\eta(v) := -\int_{S_h(\eta)} g(\eta) \cdot \bar{v} \, \mathrm{d}x. \tag{5.2}$$

To define the stochastic variation problem directly is not suitable since $V_h(\eta)$ is dependent on η. We take a transformation of variable to give a new sesquilinear form defined on $V_h \times V_h$. Let $f_0 = f(\eta_0)$ and $g_0 = g(\eta_0)$ for some fixed $\eta_0 \in \Omega$ and write $D = D(\eta_0)$, $S_h = S_h(\eta_0)$, and $V_h = V_h(\eta_0)$ for convenience. In addition, we assume that $g(\eta) \in H^1(D(\eta))^3$ and

$$\|f(\eta) - f_0\|_{1,\infty} \leq M_0 \quad \forall \eta \in \Omega$$

with some constant $M_0 > 0$. Moreover, the truncation height h is chosen such that

$$(M - m)/\gamma < 1, \tag{5.3}$$

where $\gamma = h - \sup_{x'} f_0(x')$. This condition ensures the invertiblity of the variable transform $\mathcal{H}$ which will be introduced later. Since Γ_h is artificial, choosing sufficiently large h will be enough.

Denote by $Lip(\mathbb{R}^2)$ the set of all Lipschitz continuous functions on $\mathbb{R}^2$. Then define a topological product space

$$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2,$$

where

$$\mathcal{C}_1 := \{v \in Lip(\mathbb{R}^2) : m < v < M, \|v - f_0\|_{1,\infty} \leq M_0\}$$

with constant $M_0 > 0$ and

$$\mathcal{C}_2 := H_0^1(S_h)^3.$$

The topology of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, are given by the norms $\|\cdot\|_{1,\infty}$ and $\|\cdot\|_{H^1(S_h)^3}$.

Consider the transform $\mathcal{H}$: $S_h \to S_h(\eta)$ defined by

$$\mathcal{H}(y) = y + \alpha(y_3 - f_0(y'))(f(\eta; y') - f_0(y'))e_3, \quad y \in D_h,$$

where e_3 is the unit vector in the x_3 direction and $\alpha(x)$ is a smooth cutoff function which satisfies

$$\alpha(t) = \begin{cases} 0, & t < \delta, \\ 1, & t > \gamma, \end{cases}$$

with sufficiently small $\delta > 0$. It is also required to satisfy

$$|\alpha'| < 1/(\gamma - 2\delta). \tag{5.4}$$

The Jacobi matrix of $\mathcal{H}$ is

$$\mathcal{J}_{\mathcal{H}} = I_3 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ J_1 & J_2 & J_3 \end{pmatrix},$$

where

$$J_i = \alpha(y_3 - f_0(y'))(\partial_i f(\eta; y') - \partial_i f_0(y')) - \alpha'(y_3 - f_0(y'))\partial_i f_0(y_1)(f(\eta; y') - f_0(y')), \\ i = 1, 2,$$

and

$$J_3 = \alpha'(y_3 - f_0(y'))(f(\eta; y') - f_0(y')).$$

We look for a sufficient condition for $\mathcal{J}_{\mathcal{H}}$ to be nonsingular such that $\mathcal{H}$ is invertible. According to (5.4), we let

$$|J_3| < \frac{M - m}{\gamma - 2\delta}.$$

Hence, by (5.3), we can choose δ sufficiently small such that

$$|J_3| < \frac{M - m}{\gamma - 2\delta} < 1,$$

which implies that $\mathcal{H}$ is invertible. It is easy to verify $\mathcal{H}(\Gamma_h) = \Gamma_h$. Set

$$A = (\alpha_1, \alpha_2, \alpha_3),\ B^\top = (\beta_1, \beta_2, \beta_3) \in \mathbb{C}^{3\times 3},$$

then denote

$$A : B = \mathrm{tr}(B^\top A)$$

and

$$A \otimes B = \begin{pmatrix} \alpha_2 \cdot \beta_3 - \alpha_3 \cdot \beta_2 \\ \alpha_3 \cdot \beta_1 - \alpha_1 \cdot \beta_3 \\ \alpha_1 \cdot \beta_2 - \alpha_2 \cdot \beta_1 \end{pmatrix}.$$

For $u,v\in V_h(\eta)$, taking $x=\mathcal{H}(y)$ in (5.1) yields

$$\begin{aligned}\tilde{B}_\eta(u,v) = & 2\mu\int_{S_h}\sum_{j=1}^{3}\nabla\tilde{u}_j\mathcal{J}_{\mathcal{H}^{-1}}\mathcal{J}_{\mathcal{H}^{-1}}^{\top}\nabla\bar{\tilde{v}}_j\det\mathcal{J}_{\mathcal{H}}\,\mathrm{d}y\\ &+\lambda\int_{S_h}(\nabla\tilde{u}:\mathcal{J}_{\mathcal{H}^{-1}}^{\top})(\nabla\bar{\tilde{v}}:\mathcal{J}_{\mathcal{H}^{-1}}^{\top})\det\mathcal{J}_{\mathcal{H}}\,\mathrm{d}y\\ &-\mu\int_{S_h}(J_{\mathcal{H}^{-1}}\otimes\nabla\tilde{u})(J_{\mathcal{H}^{-1}}\otimes\nabla\bar{\tilde{v}})\det\mathcal{J}_{\mathcal{H}}\,\mathrm{d}y\\ &-\omega^2\int_{S_h}\tilde{u}\cdot\bar{\tilde{v}}\det\mathcal{J}_{\mathcal{H}}\,\mathrm{d}y-\int_{\Gamma_h}\mathcal{T}\tilde{u}\cdot\bar{\tilde{v}}\,\mathrm{d}s(y),\end{aligned}$$

where $\tilde{u}=u\circ\mathcal{H}$, $\tilde{v}=v\circ\mathcal{H}$. Similarly, for $v\in V_h(\eta)$, let $x=\mathcal{H}(y)$ in (5.2),

$$\tilde{G}_\eta(v) = -\int_{D_h}\tilde{g}(\eta)\cdot\bar{\tilde{v}}\det\mathcal{J}_{\mathcal{H}}\,\mathrm{d}x.$$

Recall that we require $g(\eta)\in H^1(D(\eta))^3$ and the support of $g(\eta)$ is in $S_h(\eta)$. We have $\tilde{g}(\eta)\in H_0^1(S_h)^3$ for all η. So we can define the input map c: $\Omega\to\mathcal{C}$ by

$$c(\eta) := (f(\eta),\tilde{g}(\eta)).$$

Note that $\tilde{u},\tilde{v}\in V_h$ for $u,v\in V_h(\eta)$. Thus we can define a continuous sesquilinear form $B_{c(\eta)}(u,v)$ on $V_h\times V_h$ by

$$\begin{aligned}B_{c(\eta)}(u,v) := & 2\mu\int_{S_h}\sum_{j=1}^{3}\nabla u_j\mathcal{J}_{\mathcal{H}^{-1}}\mathcal{J}_{\mathcal{H}^{-1}}^{\top}\nabla\bar{v}_j\det\mathcal{J}_{\mathcal{H}}\,\mathrm{d}y\\ &+\lambda\int_{S_h}(\nabla u:\mathcal{J}_{\mathcal{H}^{-1}}^{\top})(\nabla\bar{v}:\mathcal{J}_{\mathcal{H}^{-1}}^{\top})\det\mathcal{J}_{\mathcal{H}}\,\mathrm{d}y\\ &-\mu\int_{S_h}(J_{\mathcal{H}^{-1}}\otimes\nabla u)(J_{\mathcal{H}^{-1}}\otimes\nabla\bar{v})\det\mathcal{J}_{\mathcal{H}}\,\mathrm{d}y\\ &-\omega^2\int_{S_h}u\cdot\bar{v}\det\mathcal{J}_{\mathcal{H}}\,\mathrm{d}y-\int_{\Gamma_h}\mathcal{T}u\cdot\bar{v}\,\mathrm{d}s(y).\end{aligned}\tag{5.5}$$

It is easy to see

$$\tilde{B}_\eta(u,v) = B_{c(\eta)}(\tilde{u},\tilde{v}).$$

Similarly we can define an antilinear functional $G_{c(\eta)}$ on V_h by

$$G_{c(\eta)}(v) := -\int_{S_h}\tilde{g}(\eta)\cdot\bar{v}\det\mathcal{J}_{\mathcal{H}}\,\mathrm{d}x.\tag{5.6}$$

Obviously, there holds the identity

$$G_{c(\eta)}(\tilde{v}) = \tilde{G}_\eta(v).$$

Then the sesquilinear form $\tilde{\mathcal{B}}$ on $L^2(\Omega;V_h)\times L^2(\Omega;V_h)$ can be defined by

$$\mathcal{B}(u,v) := \int_\Omega B_{c(\eta)}(u,v)\,\mathrm{d}\mathbb{P}(\eta)$$

and the antilinear functional $\mathcal{G}$ is defined on $L^2(\Omega; V_h)$ by

$$\mathcal{G}(v) := \int_\Omega G_{c(\eta)}(v)\,\mathrm{d}\mathbb{P}(\eta).$$

For convenience, we regard the sesquilinear form $B_{c(\eta)}\colon V_h \times V_h \to \mathbb{C}$ as the operator in $B(V_h, V_h^*)$ generated by it. Here V_h^* is the dual space of V_h and $B(X,Y)$ denotes the space including all bounded linear operators $X \to Y$. Similarly to (5.5)–(5.6), we can define the sesquilinear form $B_{(\phi,\psi)}$ and the antilinear functional $G_{(\phi,\psi)}$ for all $(\phi,\psi) \in \mathcal{C}$. Then we can define the map $\mathscr{B}\colon \mathcal{C} \to B(V_h, V_h^*)$ by

$$\mathscr{B}((\phi,\psi)) := B_{(\phi,\psi)}$$

and the map $\mathscr{G}\colon \mathcal{C} \to V_h^*$ by

$$\mathscr{G}((\phi,\psi)) := G_{(\phi,\psi)}.$$

Now we can define the stochastic variation problem as follows.

Variational problem II*:* find $u \in L^2(\Omega; V_h)$ such that

$$\mathcal{B}(u,v) = \mathcal{G}(v), \quad \forall v \in L^2(\Omega; V_h). \tag{5.7}$$

We will consider the well-posedness of the stochastic variation problem (5.7). First we show both the sesquilinear form $\mathcal{B}$ and the antilinear functional $\mathcal{G}$ are well-defined which is based on measurability and $\mathbb{P}$-essentially separability of c. For measurability and $\mathbb{P}$-essentially separability of c, the following condition is necessary.

Condition 5.1. *The map $c_1\colon \Omega \to \mathcal{C}_1$ defined by*

$$c_1(\eta) = f(\eta)$$

satisfies $c_1 \in L^2(\Omega; \mathcal{C}_1)$ and the map $c_2\colon \Omega \to \mathcal{C}_2$ defined by

$$c_2(\eta) = \tilde{g}(\eta)$$

satisfies $c_2 \in L^2(\Omega; \mathcal{C}_2)$.

We refer to [8] for the definitions of measurability and $\mathbb{P}$-essential separability. Then we give the following lemma (see Lemma 4.1 in [16]).

Lemma 5.1. *Under Condition* 5.1*, the map c is measurable and $\mathbb{P}$-essentially separable.*

Then we can prove that the sesquilinear form $\mathcal{B}$ is well-defined by the continuity of $\mathscr{B}$ and the regularity of $\mathscr{B} \circ c$.

Lemma 5.2.
- (i) *The map $\mathscr{B}\colon \mathcal{C} \to B(V_h, V_h^*)$ is continuous.*
- (ii) *The map $\mathscr{B} \circ c \in L^\infty(\Omega; B(V_h, V_h^*))$.*
- (iii) *The sesquilinear form $\mathcal{B}$ is well-defined on $L^2(\Omega; V_h) \times L^2(\Omega; V_h)$.*

Proof. We only prove (i), since (ii), (iii) can be verified similarly to the two-dimensions case in [16]. For convenience, we only prove the continuity at the point $(f_0, g_0) \in \mathcal{C}$ since for other points the proof is similar. Consider the sequence $\{(f_m, g_m)\} \subset \mathcal{C}$ such that $(f_m, g_m) \to (f_0, g_0)$ in $\mathcal{C}$ when $m \to \infty$. Denote the transform by

$$\mathcal{H}_m(y) = y + \alpha(y_3 - f_0(y'))(f_m(y') - f_0(y'))e_3, \quad y \in D_h.$$

For any $u, v \in V_h$,

$$
\begin{aligned}
&B_{(f_m,g_m)}(u,v) - B(u,v)\\
&\quad = -2\mu \int_{S_h} \sum_{j=1}^{2} \nabla u_j (I_3 - \mathcal{J}_{\mathcal{H}_m^{-1}} \mathcal{J}_{\mathcal{H}_m^{-1}}^{\top} \det \mathcal{J}_{\mathcal{H}_m}) \nabla \bar{v}_j \,\mathrm{d}x\\
&\qquad - \lambda \int_{S_h} (\nabla \cdot u)(\nabla \cdot \bar{v}) - (\nabla \tilde{u} : \mathcal{J}_{\mathcal{H}_m^{-1}})(\nabla \bar{\tilde{v}} : \mathcal{J}_{\mathcal{H}_m^{-1}}^{\top}) \det \mathcal{J}_{\mathcal{H}_m} \,\mathrm{d}x\\
&\qquad - \mu \int_{S_h} (J_{\mathcal{H}_m^{-1}} \otimes \nabla \tilde{u})(J_{\mathcal{H}_m^{-1}} \otimes \nabla \bar{\tilde{v}}) \det J_{\mathcal{H}_m} - (\nabla \times u) \cdot (\nabla \times \bar{v}) \,\mathrm{d}x\\
&\qquad - \omega^2 \int_{S_h} u \cdot \bar{v} (\det \mathcal{J}_{\mathcal{H}_m} - 1) \,\mathrm{d}x.
\end{aligned}
$$

By direct calculations, we have

$$
\det \mathcal{J}_{\mathcal{H}_m} = 1 + O(\|f_m - f_0\|_{1,\infty}), \quad \mathcal{J}_{\mathcal{H}_m^{-1}} = I_3 + O(\|f_m - f_0\|_{1,\infty}),
$$

which imply that

$$
|B_{(f_m,g_m)}(u,v) - B(u,v)| \le C \|u\|_{H^1(D_h)^2} \|v\|_{H^1(S_h)^3} \|f_m - f_0\|_{1,\infty}.
$$

For $m \to \infty$, it turns out that

$$
\|B_{(f_m,g_m)} - B\|_{B(V_h, V_h^*)} \le C \|f_m - f_0\|_{1,\infty} \to 0.
$$

This completes the proof. ∎

Next we give a similar lemma for the antilinear functional $\mathcal{G}$.

LEMMA 5.3.
(i) *The map $\mathscr{G} \colon \mathcal{C} \to V_h^*$ is continuous.*
(ii) *The map $\mathscr{G} \circ c \in L^2(\Omega; V_h^*)$.*
(iii) *The antilinear functional $\mathcal{G}$ is well-defined on $L^2(\Omega; V_h)$.*

The proof is similar to Lemma 4.3 in [16]. For any given sampling η, we consider the following deterministic variational problem III.

Find $u(\eta) \in V_h$ such that

$$
B_{c(\eta)}(u(\eta), v) = G_{c(\eta)}(v) \quad \forall v \in V_h. \tag{5.8}
$$

The existence and uniqueness of solutions of the problem (5.8) has been given in Theorem 2.1. The *a* priori bound in Theorem 4.3 can also be used for (5.8). Notice that for any η we have the upper bound

$$
L(\eta) \le L + M_0.
$$

LEMMA 5.4. *For any given η, the variational problem* (5.8) *admits a unique solution $u(\eta) \in V_h$. Moreover, the a priori bound*

$$
\|u^*(\eta)\|_{H^1(S_h(\eta))^3} \le (h - m + 2)(C_4(\omega,h) + C_5(\omega,h) + C_6(\omega,h,L_0)) \|g(\eta)\|_{H^1(S_h(\eta))^3}
$$

holds for $u^*(\eta) = u(\eta) \circ \mathcal{H}^{-1}$ with $L_0 = M_0 + L$.

Proof. If $u(\eta)$ is a solution to variational problem III (5.8), then $u^*(\eta) = u(\eta)\circ\mathcal{H}^{-1}$ is solution to variational problem I (2.17) corresponding to $f(\eta)$ and $g(\eta)$. Conversely, if $u(\eta)$ is solution to variational problem I (2.17) corresponding to $f(\eta)$ and $g(\eta)$, then $\tilde{u}(\eta) = u(\eta)\circ\mathcal{H}$ is a solution to variational Problem III (5.8). So Theorem 2.1 implies existence and uniqueness of solutions to the variational problem (5.7), and Theorem 4.3 implies the a priori bound. ∎

Lemma 5.4 shows the existence of a solution $u(\eta)$ to (5.8) for given η. In fact, the following lemma shows $u(\eta) \in L^2(\Omega; V_h)$.

LEMMA 5.5. *For the solution $u(\eta)$ to variational problem* III (5.8)*, we have $u(\eta) \in L^2(\Omega; V_h)$.*

The proof is omitted here since it is similar to the two-dimensions case in Lemma 4.4 in [16]. Based on Lemmas 5.2–5.5, we can conclude the well-posedness of (5.7) in the framework of [15, 6, 16] and extend the a priori bound to the random case as follows.

THEOREM 5.6.

(i) *variational problem* II (5.7) *admits a unique solution $u \in L^2(\Omega, V_h)$.*

(ii) *Let $u \in V_h(\eta)$ be a solution to the variational problem* I (2.17) *corresponding to $f(\eta)$ and $g(\eta)$ with $\eta \in \Omega$, and let $\tilde{u}(\eta) \in L^2(\Omega; V_h)$ be the solution to variational problem* II (5.7)*. Then u and $\tilde{u}$, respectively, satisfy the bound*

$$\int_\Omega \|u\|^2_{H^1(S_h(\eta))^3}\,\mathrm{d}\,\mathbb{P} \leq (h-m+2)^2(C_4(\omega,h)+C_5(\omega,h)+C_6(\omega,h,L_0))^2 \int_\Omega \|g\|^2_{H^1(S_h(\eta))^3}\,\mathrm{d}\,\mathbb{P},$$

and

$$\int_\Omega \|\tilde{u}\|^2_{H^1(S_h)^3}\,\mathrm{d}\,\mathbb{P} \leq (h-m+2)^2(C_4(\omega,h)+C_5(\omega,h)+C_6(\omega,h,L_0))^2 \int_\Omega \|\tilde{g}\|^2_{H^1(S_h)^3}\,\mathrm{d}\,\mathbb{P},$$

where $C_4(\omega,h), C_5(\omega,h)$, and $C_6(\omega,h,L_0)$ are given in Theorem 4.3.

6. Conclusion.

We establish the well-posedness of the time-harmonic elastic scattering from general unbounded rough surfaces in three dimensions at an arbitrary frequency. A priori bounds which are explicitly dependent on the frequency and on the geometry of the rough surface are derived both for deterministic and random cases. A possible continuation of this work is to study the elastic scattering by incident plane, spherical, or cylindrical waves. We hope to report the progress on these results in subsequent publications.

REFERENCES

[1] I. ABUBAKAR, *Scattering of plane elastic waves at rough surface I*, Math. Proc. Cambridge Philos. Soc., 58 (1962), pp. 136–157, https://doi.org/10.1017/S030500410003629X.

[2] J. D. ACHENBACH, *Wave Propagation in Elastic Solids*, North Holland, Amsterdam, 1973.

[3] T. ARENS, *The scattering of plane elastic waves by a one-dimensional periodic surface*, Math. Methods Appl. Sci., 22 (1999), pp. 55–72, https://doi.org/10.1002/(SICI)1099-1476(19990110)22:1⟨55::AID-MMA20⟩3.0.CO;2-T.

[4] T. ARENS, *Uniqueness for elastic wave scattering by rough surfaces*, SIAM J. Math. Anal., 33 (2001), pp. 461–476, https://doi.org/10.1137/S0036141099359470.

[5] T. Arens, *Existence of solution in elastic wave scattering by unbounded rough surfaces*, Math. Methods Appl. Sci., 25 (2002), pp. 507–528, https://doi.org/10.1002/mma.304.
[6] G. Bao, Y. Lin, and X. Xu, *Stability for the Helmholtz equation in deterministic and random periodic structures*, J. Math. Pures Appl. (9), 187 (2024), pp. 207–232, https://doi.org/10.1016/j.matpur.2024.05.014.
[7] S. N. Chandler-Wilde and P. Monk, *Existence, uniqueness, and variational methods for scattering by unbounded rough surfaces*, SIAM J. Math. Anal., 37 (2005), pp. 598–618, https://doi.org/10.1137/040615523.
[8] J. Diestel and B. Faires, *On vector measures*, Trans. Amer. Math. Soc., 198 (1974), pp. 253–271, https://doi.org/10.1090/S0002-9947-1974-0350420-8.
[9] J. Elschner and G. Hu, *Variational approach to scattering of plane elastic waves by diffraction gratings*, Math. Methods Appl. Sci., 33 (2010), pp. 1924–1941, https://doi.org/10.1002/mma.1305.
[10] J. Elschner and G. Hu, *Elastic scattering by unbounded rough surfaces*, SIAM J. Math. Anal., 44 (2012), pp. 4101–4127, https://doi.org/10.1137/12086203X.
[11] J. Elschner and G. Hu, *Scattering of plane elastic waves by three-dimensional diffraction gratings*, Math. Models Methods Appl. Sci., 22 (2012), 1150019, https://doi.org/10.1142/S0218202511500199.
[12] J. Elschner and G. Hu, *Elastic scattering by unbounded rough surfaces: Solvability in weighted Sobolev spaces*, Appl. Anal., 94 (2015), pp. 251–278, https://doi.org/10.1080/00036811.2014.887695.
[13] G. Hu, P. Li, and Y. Zhao, *Elastic scattering from rough surfaces in three dimensions*, J. Differential Equations, 269 (2020), pp. 4045–4407, https://doi.org/10.1016/j.jde.2020.03.022.
[14] V. D. Kupradze, et al., *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, North-Holland, Amsterdam, 1979.
[15] O. R. Pembery and E. A. Spence, *The Helmholtz equation in random media: Well-posedness and a priori bounds*, SIAM/ASA J. Uncertain. Quantif., 8 (2020), pp. 58–87, https://doi.org/10.1137/18M119327X.
[16] T. Wang, Y. Lin, and X. Xu, *A priori bounds for elastic scattering by deterministic and random unbounded rough surfaces*, CSIAM Trans. Appl. Math., 4 (2023), pp. 696–720, https://doi.org/10.4208/csiam-am.SO-2023-0001.