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Radiation conditions for the Helmholtz equation in a half plane filled by inhomogeneous periodic material

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Abstract

In this paper we consider time-harmonic acoustic wave propagation in a half-plane filled by inhomogeneous periodic medium. If the refractive index depends on the horizontal coordinate only, we define upward and downward radiating modes by solving a one-dimensional Sturm-Liouville eigenvalue problem with a complex-valued periodic coefficient. The upward and downward radiation conditions are introduced based on a generalized Rayleigh series. Using the variational method, we then prove uniqueness and existence for the scattering of an incoming wave mode by a grating located between an upper and lower half plane with such inhomogeneous periodic media. The solution operator of the quasiperiodic boundary value problem yields a well-defined scattering operator, the numerical approximation of which is nothing else but the S-matrix in scattering matrix algorithms like Rigorous Coupled Wave Analysis (RCWA) or Fourier Modal Method (FMM).

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1. Introduction

Since Lord Rayleigh's original work [29] in 1907, time harmonic scattering problems by periodic and even by biperiodic gratings are well studied in both the physical and mathematical communities. The theory provides a Rayleigh expansion radiation condition over the half plane filled by homogeneous material. Using this, the acoustic, elastic and electromagnetic diffraction problems have been studied extensively concerning theoretical analysis and numerical approximation using integral equation and variational methods (cf. e.g. [1,3,4,7,8,10–12,24,32,37,38]). We refer to [5,33–35] for historical remarks and details of engineering applications, if the cover material in the half spaces above and the substrate material below the periodic surface structure of the grating is supposed to be homogeneous. However, special inhomogeneous materials are possible in applications. For instance, in the design of photonic crystals, the refractive index corresponding to materials of interest is a periodic function in different spatial directions. This paper is devoted to new radiation conditions for the Helmholtz equation and the corresponding solvability theory. The theory applies to the analysis of the scattering matrix algorithm even for the solution of classical scattering problems with homogeneous cover and substrate material.

To start the analysis, we consider the case of periodic gratings in the two-dimensional space contained in the layer $\{(x_1, x_2)^\top \in \mathbb{R}^2 : b \leq x_2 \leq d\}$, where the refractive index $(x_1, x_2)^\top \mapsto \mathrm{ind}(x_1)$ in the half planes $\{(x_1, x_2)^\top \in \mathbb{R}^2 : d \leq x_2\}$ of cover material and $\{(x_1, x_2)^\top \in \mathbb{R}^2 : x_2 \leq b\}$ of substrate material is independent of the vertical x_2 and a periodic function with respect to the horizontal x_1. We assume $\mathrm{ind}(x_1 + p) = \mathrm{ind}(x_1)$ with the same period p as for the grating structure. Similarly to the homogeneous case, the radiation condition for these half planes is defined by expansions into a generalized Rayleigh series of upgoing and downgoing wave modes. If the refractive index is real-valued, we need to analyze an infinite-dimensional ordinary differential equation by the spectral theorem for self-adjoint operators. In the general case of complex-valued potentials, the resulting system is no longer self-adjoint. Instead, we consider a linear 2-by-2 ODE system that is equivalent to the Helmholtz equation in two dimensions. The solutions of the ODE system are connected to those of a non-selfadjoint Sturm-Liouville differential operator. The wave modes in this case take the form $(x_1, x_2)^\top \mapsto \exp(\lambda x_2)h(x_1)$, where λ is an eigenvalue and h an eigenfunction or a linear combination of associated eigenfunctions of the Sturm-Liouville differential operator. These functions can be classified into outgoing upward and downward wave modes depending on the sign of λ, giving rise to the radiation conditions as $x_2 \to \pm\infty$. Using these natural conditions, we can defined the Dirichlet-to-Neumann (DtN) map on an artificial boundary to truncate the unbounded lower half-plane to a bounded domain in a single periodic cell. We show the properties of the DtN map over Sobolev spaces. Then we verify the Fredholm property for the boundary value problem modeling the scattering of an incoming wave mode by the grating. Uniqueness is shown for the propagating reflected and transmitted wave modes. The full solution is proved to be unique if the grating contains absorbing materials.

Our research is closest to the recent work [27], where a technical outgoing radiation condition was proposed to analyze the transmission problem between free space and an unbounded photonic crystal. In comparison with [27], we assume that the inhomogeneous material is invariant along the vertical coordinate x_2, leading to more explicit upward and downward radiating modes and stronger uniqueness and existence results. The methodology used in this work differs from other scattering problems arising from closed periodic waveguides [14] (see also [13]), infinite periodic cylinders [26] and in stratified media [25], which rely essentially on Floquet-Bloch transform and the limiting absorption principle. The contributions of [13,25,26] focus on radiation conditions of the Helmholtz equation caused by a compactly supported source term in a closed or open waveguide, and the approach of using Floquet-Bloch transform results in a set of quasiperiodic boundary value problems depending on the parameters of quasiperiodicity (phase-shifts). However, in our diffraction problem, the quasiperiodic parameter arises from incident waves without compact support and is fixed throughout the whole paper. The classification of left-going and right-going wave modes in [13,25,26] relies on the assumption of non-vanishing group velocities for dispersion curves, while the upward and downward radiation conditions within this paper are defined under Assumption RC(q) (see Definition 4.15). The materials in the aforementioned works are usually assumed to be periodic inside the waveguide and to be identical in the exterior, whereas in our settings, the inhomogeneous periodic material occupies a half plane. All of these different radiation conditions extend the classical Rayleigh Expansion Radiation Condition from infinitely homogeneous periodic settings to more complicated periodic materials. We also refer to [2,6,19,36] for earlier studies on radiating modes in open and semi-infinite waveguides.

It should be remarked that the new radiation conditions proposed in this paper can be used to analyze the scattering matrix algorithm, one of the most popular numerical methods for the classical periodic gratings. Its various versions are called Rigorous Coupled Wave Analysis (RCWA) or Fourier Modal Method (FMM) (cf. e.g. [17,18,28,31,33,34]). Unfortunately, there is little analysis available so far. The technique of ODEs is difficult to apply since differential operators with piecewise constant coefficients act on the horizontal functions. Instead, the spaces and theorems for the Helmholtz equations should be used. We refer to the original preprint [23, Sect. 6] of this paper for discussions on the application of our theory to the scattering matrix algorithm.

This paper is organized as follows. We introduce the inhomogeneous half spaces with cover and substrate material as well as the corresponding boundary value problems in Sect. 2. In Sect. 3, supposing non-absorbing materials, we define the radiation condition by Fourier series expansion with respect to x_1 and by solving a function valued ODE with techniques of functional analysis. Alternatively, we solve the ODE with operator valued coefficient by an eigenvalue decomposition for this coefficient operator acting on quasiperiodic functions with respect to x_1. A general periodic medium (including absorbing material) will be investigated in the Subsects. 4.2 and 4.3, where we discuss the eigenvalues, eigenfunctions and associated eigenfunctions for the coefficient operator in Sturm-Liouville form. This decomposition is used to define upward and downward radiating wave modes and the radiation condition in Subsect. 4.4. In Sect. 5 we introduce the boundary value problem for gratings between an upper and lower half space of inhomogeneous media. We present the variational formulation and discuss the uniqueness and existence of weak solutions.

2. Quasiperiodic boundary value problem in an inhomogeneous half space

Denoting the points in two-dimensional space by $x = (x_1, x_2)^\top$, we suppose that the lower half space $\Omega_b^- := \{x \in \mathbb{R}^2 : x_2 < b\}$ is illuminated by an incoming wave from the upper half space $\Omega_b^+ := \{x \in \mathbb{R}^2 : x_2 > b\}$ with the wave number $k > 0$. In this paper it is assumed that Ω_b^- is occupied by an inhomogeneous periodic medium modeled by the squared refractive index (potential) $q \in L^\infty(\Omega_b^-)$ (cf. Fig. 1). Further, q is assumed to be independent of x_2 and 2π-periodic in x_1, i.e.,

$$q(x) = q(x_1),\ q(x_1 + 2\pi n) = q(x_1) \qquad \text{for a.e. } x_1 \in \mathbb{R} \text{ and all } n \in \mathbb{Z}. \tag{2.1}$$

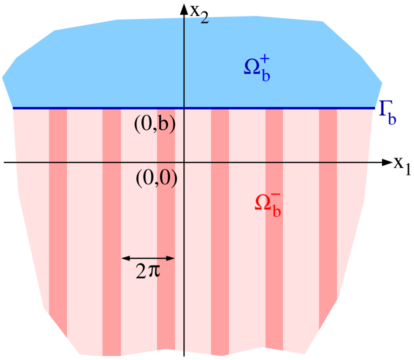

Fig. 1. The geometry settings.

For physical reasons, we suppose that there is a $c_q > 0$ such that either $q(x_1) \geq c_q$ or $\operatorname{Im} q(x_1) \geq c_q$ for a.e. $x_1 \in \mathbb{R}$.

Then the time-harmonic acoustic wave propagation in Ω_b^- is governed by the Helmholtz equation $\Delta u + k^2 q u = 0$ in Ω_b^-, where $u = u(x)$ denotes the acoustic pressure or a component of an electromagnetic field. Since the lower half space in unbounded, we need a radiation condition of u as x_2 tends to $-\infty$ to ensure well-posedness of the scattering problem. To mathematically formulate the scattering problem, we need the concept of quasiperiodic functions and Sobolev spaces.

Definition 2.1. The function u is called quasiperiodic in x_1 with the parameter $\alpha \in [0, 1)$ (that is, α-quasiperiodic), if $x_1 \mapsto u(x_1, x_2)e^{-i\alpha x_1}$ is 2π-periodic in x_1 for any fixed x_2.

Clearly, α-periodic functions satisfy the relation

$$u(x_1 + 2n\pi, x_2) = e^{i2n\pi\alpha} u(x_1, x_2) \quad \text{for all } n \in \mathbb{Z}. \tag{2.2}$$

Define the quasiperiodic Sobolev spaces on Ω_b^- and $\mathbb{R}$ by

$$\begin{aligned} H_\alpha^1(\Omega_b^-) &:= \{u \in H_{loc}^1(\Omega_b^-) : u \text{ is } \alpha\text{-quasiperiodic in } x_1\} \\ H_\alpha^{1/2}(\mathbb{R}) &:= \{f \in H_{loc}^{1/2}(\mathbb{R}) : e^{-i\alpha x_1} f(x_1) \text{ is } 2\pi\text{-periodic in } x_1\}. \end{aligned}$$

Note that our $H_{loc}^1(\Omega_b^-)$ is the space of all functions u over Ω_b^- such that, for any radius $r > 0$, the restriction of u to $\Omega_{b,r}^- := \{x \in \Omega_b^- : |x| < r\}$ is in $H^1(\Omega_{b,r}^-)$. If the incoming wave is a plane wave of the form $u^{in}(x) := \exp(ik(x_1 \sin\theta - x_2 \cos\theta))$ with the incident angle $\theta \in (-\pi/2, \pi/2)$, we set $\alpha_0 := k \sin\theta$ and get an α-quasiperiodic function u^{in} with α the unique number such that $\alpha \in [0, 1)$ and $\alpha - \alpha_0$ is an integer (see (2.2)). In the case $q \equiv 1$ in Ω_b^-, we recall that a Helmholtz solution u is called downward radiating if u admits a Rayleigh expansion (see, e.g., [1,12,24])

$$u(x) = \sum_{n\in\mathbb{Z}} c_n \, e^{i(\alpha_n x_1 - \beta_n (x_2 - b))}, \qquad x_2 < b, \tag{2.3}$$

where the $c_n \in \mathbb{C}$ are called Rayleigh coefficients and

$$\alpha_n := n + \alpha_0, \quad \beta_n := \begin{cases} \sqrt{k^2 - \alpha_n^2} & \text{if} \quad |\alpha_n| \le k, \\ i\,\sqrt{\alpha_n^2 - k^2} & \text{if} \quad |\alpha_n| > k. \end{cases} \tag{2.4}$$

The existence of coefficients c_n with Equ. (2.3) is called the radiation condition for the lower half plane Ω_b^-. The upward radiation condition in Ω_b^+ filled by a homogeneous medium can be defined analogously. Obviously, the Rayleigh expansion (2.3) consists of a finite number of propagating waves corresponding to n with $|\alpha_n| \le k$ and an infinite number of evanescent waves for $|\alpha_n| > k$, which decay exponentially when $|x_2| \to \infty$. It has been widely used in the literature to prove well-posedness and design numerical schemes for time-harmonic acoustic, elastic and electromagnetic scattering by periodic surface structures located between half spaces occupied by homogeneous media [1,3,4,7,8,10–12,21,22,24,32]. One of the main subjects of the present paper is to define downward and upward radiation conditions in an inhomogeneous medium, which will generalize the above Rayleigh expansion from a homogeneous periodic medium to the inhomogeneous case of (2.1).

Consider the boundary value problem in an inhomogeneous half space

$$\text{(BVP):} \quad \begin{cases} \Delta u + k^2 q u = 0 & \text{in} \quad \Omega_b^-, \\ u = f & \text{on} \quad \Gamma_b := \left\{x \in \mathbb{R}^2 : x_2 = b\right\}, \end{cases} \tag{2.5}$$

where $f \in H_\alpha^{1/2}(\mathbb{R})$. We shall define an ‘appropriate’ downward radiation condition over Ω_b^- and prove, under some additional assumptions, that the boundary value problem (2.5) combined with the radiation condition has a unique solution $u \in H_\alpha^1(\Omega_b^-)$ for any given $f \in H_\alpha^{1/2}(\Gamma_b)$.

3. Radiation condition for real-valued potentials

In this section we suppose that the squared refractive index function q with $q(x) = q(x_1)$ and with $q \in L^\infty(0, 2\pi)$ is real-valued. Now we shall show that the Helmholtz equation is equivalent to an ODE in the space of sequences of Fourier coefficients. Trying to get a Rayleigh expansion in an inhomogeneous medium, we first look at the Fourier expansion of the solution. Since u is α-quasiperiodic, it admits the expansion

$$e^{-i\alpha x_1} u(x_1, x_2) = \sum_{n\in\mathbb{Z}} u_n(x_2) e^{inx_1}, \quad x_2 < b,$$

or equivalently,

$$u(x_1, x_2) = \sum_{n\in\mathbb{Z}} u_n(x_2) e^{i\alpha_n x_1}, \quad x_2 < b. \tag{3.1}$$

Inserting (3.1) into the Helmholtz equation, we find that

$$\sum_{n\in\mathbb{Z}} \left[u_n''(x_2) + \left(k^2 q(x_1) - \alpha_n^2\right) u_n(x_2)\right] e^{i\alpha_n x_1} = 0 \quad \text{for all} \quad x_1 \in \mathbb{R}. \tag{3.2}$$

For constant $q \equiv 1$ this leads to a separate ordinary differential equation for each coefficient function u_n, to the solution $e^{i(\alpha_n x_1 - \beta_n(x_2-b))}$, and finally to the classical Rayleigh expansion. Unfortunately, if $q(x_1)$ depends on x_1, then we cannot replace the Rayleigh modes $e^{i(\alpha_n x_1 - \beta_n(x_2-b))}$ in (2.3) by $e^{i\alpha_n x_1} u_n(x_2)$ with u_n the solution of a second-order ODE.

In order to introduce norms for the trace of the solution to the boundary value problem (2.5), we may expand the Dirichlet data $f = u|_{\Gamma_b}$ into the Fourier series

$$f(x_1) = \sum_{n\in\mathbb{Z}} f_n\, e^{i\alpha_n x_1}, \quad f_n \in \mathbb{C}.$$

We introduce the weighted ℓ^2 space of sequences

$$X^s := \left\{ \mathbf{a} = (a_n)_{n\in\mathbb{Z}} : \sum_{n\in\mathbb{Z}} (1+n^2)^s |a_n|^2 < \infty \right\}$$

endowed with the inner product and norm

$$\langle \mathbf{a}, \mathbf{b} \rangle_s := \sum_{n\in\mathbb{Z}} (1+n^2)^s a_n\, \overline{b}_n, \quad \|a\|_{X^s} := \sqrt{\sum_{n\in\mathbb{Z}} (1+n^2)^s |a_n|^2}.$$

Then X^s is a Hilbert space for any $s \in \mathbb{R}$. The Fourier coefficients of f satisfy

$$\|f\|_{H^{1/2}_\alpha(\Gamma_b)} = \|\mathbf{f}\|_{X^{1/2}} < \infty, \quad \mathbf{f} := (f_n)_{n\in\mathbb{Z}}.$$

Applying Fourier expansion to the refractive index function, we have

$$q(x_1) = \sum_{m\in\mathbb{Z}} q_m e^{imx_1}, \quad q_m \in \mathbb{C}. \tag{3.3}$$

Obviously, we would have $q \equiv q_0$ if the medium of Ω_b^- is homogeneous. Inserting the above expansion into (3.2), it follows that

$$\sum_{n\in\mathbb{Z}} \left[\Big(u_n''(x_2) - \alpha_n^2 u_n(x_2) \Big) e^{i\alpha_n x_1} + k^2 \sum_{m\in\mathbb{Z}} q_m e^{i\alpha_{n+m} x_1} u_n(x_2) \right] = 0,\ x \in \Omega_b^-.$$

Multiplying the previous equation by $e^{-i\alpha_j x_1}$ and integrating over $(0, 2\pi)$ with respect to x_1 lead to

$$u_j'' - \alpha_j^2 u_j + k^2 \sum_{m\in\mathbb{Z}} q_{j-m} u_m = 0, \quad j \in \mathbb{Z}.$$

We set $U(x_2) := (\cdots, u_{-1}(x_2), u_0(x_2), u_1(x_2), \cdots)$. Since the function $x_1 \mapsto u(x_1, x_2)$ is in $H^{1/2}_\alpha(\mathbb{R})$ for any $x_2 \le b$, it holds that $U(x_2) \in X^{1/2}$ for any fixed $x_2 \le b$. The previous equations can be rewritten as a second-order ODE in the form

$$U''(x_2) + A\, U(x_2) = 0, \quad x_2 < b, \tag{3.4}$$

where $A := (a_{jm})_{j,m\in\mathbb{Z}}$ is an infinite dimensional matrix, whose entries are given by

$$a_{jm} := \begin{cases} k^2 q_{j-m} & \text{if} \quad j \neq m, \\ -\alpha_j^2 + k^2 q_0 & \text{if} \quad j = m. \end{cases}$$

The matrix A can be written as $A = B + k^2 C$, where $B := (b_{j,m})_{j,m\in\mathbb{Z}}$ is the diagonal matrix and $C := (c_{j,m})_{j,m\in\mathbb{Z}}$ the Toeplitz matrix defined by

$$b_{j,m} := \begin{cases} 0 & \text{if} \quad j \neq m, \\ -\alpha_j^2 & \text{if} \quad j = m. \end{cases} \qquad c_{j,m} := q_{j-m}.$$

Evidently, the operator $B : X^{1/2} \to X^{-1/2}$ is bounded. The embedding theorems together with the fact that $q \in L^\infty(0, 2\pi)$ imply that the operator $C : X^{1/2} \to X^{-1/2}$ is compact. Since q is real-valued, we have $q_m = \bar{q}_{-m}$. It then follows that the matrix $A : X^{1/2} \to X^{-1/2}$ is a linear self-adjoint operator. Moreover, the spectrum $\sigma(A)$ of A is real.

Now the solution of the ODE (3.4) follows the classical theory of linear ODEs with constant coefficients. By the spectral theorem, we may express A as an integral over the spectrum with respect to a projection-valued measure, that is,

$$A = \int_{\sigma(A)} \lambda \, dP_\lambda.$$

For simplicity assume that $0 \notin \sigma(A)$. We define $\chi_{\mathbb{R}^\pm} : \mathbb{R} \to \mathbb{R}$ to be the characteristic function of the half line $\mathbb{R}^\pm$ and

$$A^\pm := \int_{\sigma(A)} \chi_{\mathbb{R}^\pm}(\lambda)\lambda \, dP_\lambda, \quad \sqrt{A^\pm} := \int_{\sigma(A)} \chi_{\mathbb{R}^\pm}(\lambda)\sqrt{\pm\lambda} \, dP_\lambda.$$

Evidently, we have $A = A^+ + A^-$ and $\sqrt{A} = \sqrt{A^+} + i\sqrt{A^-}$. The general solution to (3.4) is of the form

$$\begin{aligned} U(x_2) &= e^{i\sqrt{A}x_2}\mathbf{a}^+ + e^{-i\sqrt{A}x_2}\mathbf{a}^- \\ &= (e^{i\sqrt{A^+}x_2} + e^{-\sqrt{A^-}x_2})\mathbf{a}^+ + (e^{-i\sqrt{A^+}x_2} + e^{\sqrt{A^-}x_2})\,\mathbf{a}^- \end{aligned} \tag{3.5}$$

with $\mathbf{a}^\pm \in X^{1/2}$ and with $e^{\pm i\sqrt{A}x_2}$ to be understood as the exponential of an operator. In fact, straightforward calculations show that

$$\begin{aligned} (e^{i\sqrt{A^\pm}x_2}\mathbf{a}^\pm)'' = -A^\pm e^{i\sqrt{A^\pm}x_2}\mathbf{a}^\pm &= \int_{\sigma(A)} -\chi_{\mathbb{R}^\pm}(\lambda)\,\lambda e^{i\sqrt{\pm\lambda}x_2} \, dP_\lambda \, \mathbf{a}^\pm \\ &= \int_{\sigma(A)} -\lambda \, dP_\lambda \int_{\sigma(A)} \chi_{\mathbb{R}^\pm}(\lambda) e^{i\sqrt{\pm\lambda}x_2} \, dP_\lambda \, \mathbf{a}^\pm \\ &= -A \, e^{i\sqrt{A^\pm}x_2} \, \mathbf{a}^\pm. \end{aligned}$$

This implies that

$$U'' = (e^{i\sqrt{A^+}x_2}\mathbf{a}^+)'' + (e^{i\sqrt{A^-}x_2}\mathbf{a}^-)'' = -Ae^{i\sqrt{A^+}x_2}\mathbf{a}^+ - Ae^{i\sqrt{A^-}x_2}\mathbf{a}^- = -AU,$$

which proves that the function $U(x_2)$ given by (3.5) is a solution of the infinite dimensional system (3.4). Since u should be downward radiating, we require u not to contain upgoing plane waves $e^{i\sqrt{A^+}x_2}\mathbf{a}^+$ and to be bounded for $x_2 < b$, i.e., $\mathbf{a}^+ \equiv 0$. Recalling $u|_{\Gamma_b} = f$, it follows from (3.5) that $\mathbf{a}^- = e^{i\sqrt{A}b}\mathbf{f}$, $\mathbf{f} := (f_n)_{n\in\mathbb{Z}}$. This implies that

$$U(x_2) = e^{-i\sqrt{A}(x_2-b)}\mathbf{f}.$$

Definition 3.1. If $q(x) = q(x_1)$ and $q \in L^\infty(0, 2\pi)$ is real-valued, then $u \in H^1_\alpha(\Omega_b^-)$ is said to be a downward radiating solution to the Helmholtz equation if

$$u(x_1, x_2) = \sum_{n\in\mathbb{Z}} \left[e^{-i\sqrt{A}(x_2-b)}\mathbf{g}\right]_n e^{i\alpha_n x_1}, \quad x_2 \le b,$$

for some $\mathbf{g} \in X^{1/2}$. Here the notation $[\cdot]_n$ stands for the nth entry of an infinite dimensional vector.

The upward radiation condition in $x_2 \ge b$ can be defined analogously by replacing $-i\sqrt{A}$ with $i\sqrt{A}$. The above downward radiation condition allows us to express the solution to the boundary value problem (2.5) as

$$u(x_1, x_2) = \sum_{n\in\mathbb{Z}} \left[e^{-i\sqrt{A}(x_2-b)}\mathbf{f}\right]_n e^{i\alpha_n x_1}, \quad x_2 \le b.$$

Remark 3.2. If $q \equiv q_0 = 1$, all the off-diagonal terms of A vanish and the diagonal terms take the form $a_{nn} = k^2 - \alpha_n^2$ for all $n \in \mathbb{Z}$. This implies that $(\sqrt{A})_{nn} = \beta_n$, where $\beta_n \in \mathbb{C}$ is defined in (2.4). Hence, we have

$$\left[e^{-i\sqrt{A}(x_2-b)}\mathbf{f}\right]_n = e^{-i\beta_n(x_2-b)} f_n,$$

that is, u takes the same form as (2.3). The new radiation condition in Definition 3.1 is a generalization of the classical radiation condition for periodic gratings with homogeneous cover and substrate material.

We remark that the real-valued bounded index function q gives rise to a self-adjoint operator A and particularly excludes eigenvalues with generalized (associated) eigenfunctions in the spectrum of A. This has significantly simplified the arguments in comparison to the complex-valued potentials, which will be presented in Sect. 4 below. It is possible to define an equivalent Dirichlet-to-Neumann map to the downward radiating condition of Definition 3.1 and then prove Fredholm property of the resulting variational formulation in one periodic cell. We omit the details, since a more general framework will be presented in Sect. 4. However, this section has its own interests for investigating the x_1-dependent real-valued potential, in particular when the expansion (3.3) has a finite number of non-vanishing Fourier coefficients.

4. Radiation condition for complex-valued potentials

Assume that $q(x) = q(x_1)$, where $q \in L^\infty(0, 2\pi)$ is complex-valued. We shall derive a generalized Rayleigh expansion into wave modes of the form $e^{\lambda x_2} h(x_1)$ instead of the $e^{i(\alpha_n x_1 - \beta_n(x_2-b))}$ in (2.3) or the $e^{i\alpha_n x_1} u_n(x_2)$ in (3.1). The functions h will be quasiperiodic eigenfunctions of a special ODE with respect to x_1, and the λ will be the corresponding eigenvalues. We shall consider the Helmholtz equation in Ω_b^- as a second-order ODE with respect to $x_2 \in (\infty, b)$, where the solution takes the function $\mathbb{R} \ni x_1 \mapsto u(x_1, x_2)$ as values at x_2. As usually, the second-order ODE is equivalent to a linear first-order 2-by-2 ODE system. The coefficient M, an ordinary differential operator with respect to x_1, is independent of x_2. Using the eigenvalues and generalized eigenfunctions of M, we can represent any solution as a Rayleigh series of wave modes, where, roughly speaking, each mode is the product of a generalized eigenfunction depending on x_1 times an exponential $e^{\lambda x_2}$ with λ the eigenvalue. In other words, in this section we write the Helmholtz equation as a linear second-order ODE with constant operator coefficient L. In Subsect. 4.1 we shall derive the equivalent first-order ODE with operator coefficient M. This 2-by-2 operator contains L in one of its entries. We shall analyze eigenvalues and eigenfunctions for L and M and special wave modes in Subsects. 4.2 and 4.3. Finally, we shall define the wave modes for the Rayleigh series and the radiation conditions in Subsect. 4.4.

4.1. *Ordinary differential equation with respect to x_1*

To get an equivalent first-order ODE, we set $\partial_j u = \partial u / \partial x_j$ $(j = 1, 2)$, $v := \partial_2 u$, and $W := (u, v)^\top$. Clearly, introducing the second-order ordinary differential operator

$$(Lf)(x_1) := -\frac{\mathrm{d}^2 f(x_1)}{\mathrm{d}x_1^2} - k^2 q(x_1) f(x_1), \tag{4.1}$$

the Helmholtz equation $(\Delta + k^2 q I)u = 0$ is equivalent to the function-valued second-order ODE $\partial_2^2 u(\cdot, x_2) - Lu(\cdot, x_2) = 0$, or equivalently, $\partial_2 v = Lu$. Hence, the Helmholtz equation can be written in the matrix-vector form

$$\partial_2 W = M\, W, \quad M := \begin{pmatrix} 0 & I \\ L & 0 \end{pmatrix}. \tag{4.2}$$

The domain of L is defined as

$$\mathcal{D} := \Big\{ f \in L^2(0, 2\pi) : f, f' \text{ are absolute continuous and } \alpha\text{-quasiperiodic}, Lf \in L^2(0, 2\pi) \Big\}.$$

Note that L is self-adjoint over $\mathcal{D}$ if and only if the potential q is real-valued. It is well-known that the spectrum of L is purely discrete. In the Subsects. 4.2 and 4.3 we shall investigate the relation between the spectra of M and L. The eigenvalues and associated eigenfunctions of L and M are defined as follows.

Definition 4.1. A number $\lambda \in \mathbb{C}$ is called an eigenvalue of the differential operator M combined with α-quasiperiodic boundary conditions, if the α-quasiperiodic boundary value problem $MW = \lambda W$ has at least one nontrivial solution $W = (w, v)^\top \in \mathcal{D}^2$. The function W is called eigenfunction corresponding to λ. Furthermore, we define associated eigenfunction of rank

$m \geq 1$ by induction. A function $W \in \mathcal{D}^2$ is called associated eigenfunction of rank one of M corresponding to λ if it is an eigenfunction corresponding to λ. For $m > 1$, a function $W \in \mathcal{D}^2$ is called associated eigenfunction of rank m of M corresponding to λ if $W' := (M - \lambda \mathbf{I})\, W$ is a nontrivial associated eigenfunction of rank $m-1$ corresponding to λ. Here $\mathbf{I}$ denotes the 2-by-2 identity matrix. The functions $W^{(j)} := (M - \lambda\mathbf{I})^j W$ with $j \geq 0$ and $W^{(0)} := W$ will be referred to as the chain of associated eigenfunctions generated by W. The couple (λ, W) is called an eigenpair of the differential operator M if W is an associated eigenfunction of rank $m \geq 1$ corresponding to the eigenvalue λ.

Definition 4.2. A number $\mu \in \mathbb{C}$ is called an eigenvalue of the differential operator L combined with α-quasiperiodic boundary conditions, if the α-quasiperiodic boundary value problem $Lh = \mu h$ has at least one nontrivial solution $h \in \mathcal{D}$. The function h is called eigenfunction corresponding to μ. Furthermore, we define associated eigenfunction of rank $m \geq 1$ by induction. A function $h \in \mathcal{D}$ is called associated eigenfunction of rank one of L corresponding to μ if it is an eigenfunction of L corresponding to μ. For $m > 1$, a function $h \in \mathcal{D}$ is called associated eigenfunction of rank m of L corresponding to μ if the function $h^{(1)} := (L - \mu I)\, h$ is a nontrivial associated eigenfunction of rank $m-1$ corresponding to μ. The functions $h^{(j)} := (L - \mu I)^j h$ with $j \geq 0$ and $h^{(0)} := h$ will be referred to as the chain of associated eigenfunctions generated by h. The couple (μ, h) is called an eigenpair of the differential operator L if h is an associated eigenfunction of rank $m \geq 1$ corresponding to the eigenvalue μ.

We conclude this subsection presenting an example of eigenvalues and eigenfunctions for L, where $k = 1$ and q is a piecewise constant function. Note that this α-quasiperiodic eigenvalue problem is a special case of the Sturm-Liouville eigenvalue problem for the second order ordinary differential operator L over the interval $(0, 2\pi)$. For the proofs we refer to the techniques in [30]. We fix numbers $q_j \in \mathbb{C},\ j = 0, 1$ and consider the squared refractive-index function

$$q(x_1) := \begin{cases} q_0/k^2 & \text{if } 0 < x_1 < \pi \\ q_1/k^2 & \text{if } \pi < x_1 < 2\pi \end{cases}.$$

If μ is sufficiently large, then there are no associated eigenfunctions of rank greater one. For an eigenvalue μ, the eigenfunction h is given by

$$h(x_1) := \begin{cases} a\dfrac{\sin\left(\sqrt{q_0+\mu}\, x_1\right)}{\sqrt{q_0+\mu}} + \cos\left(\sqrt{q_0+\mu}\, x_1\right) & \text{if } 0 < x_1 < \pi \\ e^{\mathrm{i}\alpha 2\pi}\left\{ a\dfrac{\sin\left(\sqrt{q_1+\mu}\,(x_1-2\pi)\right)}{\sqrt{q_1+\mu}} + \cos\left(\sqrt{q_1+\mu}\,(x_1-2\pi)\right)\right\} & \text{if } \pi < x_1 < 2\pi \end{cases}, \tag{4.3}$$

$$a := e^{\mathrm{i}\alpha 2\pi} \cos\left(\sqrt{q_1+\mu}\,\pi\right) - \cos\left(\sqrt{q_0+\mu}\,\pi\right) = h'(0).$$

Note that it does not matter which sign for the square root $\sqrt{q_0+\mu}$ and $\sqrt{q_1+\mu}$ is taken. Clearly, the formula (4.3) for h requires $\sqrt{q_j+\mu} \neq 0$ for $j = 1, 2$. If $\sqrt{q_j+\mu} = 0$, then we define the quotient $\sin(\sqrt{q_j+\mu}\, x_1)/\sqrt{q_j+\mu} := x_1$, and the formula remains true. From the definition of h and a, it is clear that h is α-periodic in x_1. The eigenvalues are those μ for which $h(\pi+0) = h(\pi-0)$ and $h'(\pi+0) = h'(\pi-0)$. Simple calculations show that

Table 1
First ten eigenvalues for the case $\alpha = 0$, $q_0 = 1$, and $q_1 = 2$.

j	asymptotics of $\mu_{j,\pm}$	$\mu_{j,+}$	$\mu_{j,-}$
1	-0.43750	-0.51990	-0.36619
2	2.51562	2.4851	2.5457
3	7.50694	7.4901	7.5237
4	14.50391	14.493	14.515
5	23.50250	23.494	23.512
6	34.50174	34.501	34.502
7	47.50128	47.501	47.502
8	62.50098	62.501	62.501
9	79.50077	79.501	79.501
10	98.50062	98.501	98.501

$$\begin{aligned}
h(\pi+0) &= e^{\mathbf{i}\alpha 2\pi}\left\{-a\frac{\sin\left(\sqrt{q_1+\mu}\,\pi\right)}{\sqrt{q_1+\mu}} + \cos\left(\sqrt{q_1+\mu}\,\pi\right)\right\},\\
h'(\pi+0) &= e^{\mathbf{i}\alpha 2\pi}\left\{-a\cos\left(\sqrt{q_1+\mu}\,\pi\right) + \sqrt{q_1+\mu}\sin\left(\sqrt{q_1+\mu}\,\pi\right)\right\},\\
h(\pi-0) &= a\frac{\sin\left(\sqrt{q_0+\mu}\,\pi\right)}{\sqrt{q_0+\mu}} + \cos\left(\sqrt{q_0+\mu}\,\pi\right),\\
h'(\pi-0) &= a\cos\left(\sqrt{q_0+\mu}\,\pi\right) - \sqrt{q_0+\mu}\sin\left(\sqrt{q_0+\mu}\,\pi\right).
\end{aligned}$$

From this and from the continuity of h and h' at $x_1 = \pi$, it is not hard to deduce that the eigenvalues are the zeros of the function

$$\begin{aligned}
\det(\mu) := -1 - e^{\mathbf{i}\alpha 4\pi} + 2e^{\mathbf{i}\alpha 2\pi}\cos(\sqrt{q_0+\mu}\,\pi + \sqrt{q_1+\mu}\,\pi)\\
-e^{\mathbf{i}\alpha 2\pi}\frac{\sin(\sqrt{q_0+\mu}\,\pi)\sin(\sqrt{q_1+\mu}\,\pi)}{4(\sqrt{q_0+\mu}+\sqrt{q_1+\mu})^2\sqrt{q_0+\mu}\sqrt{q_1+\mu}}.
\end{aligned}$$

Similarly to the derivation of the general asymptotics (cf. Lemma 4.5 (iv)), we arrive at the special asymptotics (cf. a special case in Table 1)

$$\mu_{j,\pm} := (j \pm \alpha)^2 - \frac{q_0+q_1}{2} + O\left(|j|^{-\kappa}\right),\ |j| \to \infty,\ j \in \mathbb{Z}.$$

Here we have $\kappa := 1.5$ for $\alpha \neq 1/2$ and $\kappa := 0.5$ else. Indeed, neglecting the last term on the right-hand side, the main part of the asymptotics follows easily. A standard perturbation argument proves the estimate of the remainder. Moreover, it is not hard to see that $\mu_{j,+} \neq \mu_{j,-}$ for sufficiently large $|j|$.

4.2. Spectra of non-zero eigenvalues

Supposing that $\mu \in \mathbb{C}$ is a non-zero eigenvalue of L, we shall present two linearly independent solutions to the boundary value problem (BVP) (see (2.5)) using the eigenspace corresponding to

μ. To make the solutions physically meaningful, we need additional assumptions on q (or L). The case of $\mu = 0$ will be investigated in the subsection 4.3. For clarity, we divide this subsection into three parts. Firstly, the spectra of the 2-by-2 matrix operator M will be derived from the spectra of L. Secondly, it will be discussed, whether the eigenfunctions and associated eigenfunctions of L form a Riesz basis of $L^2(0, 2\pi)$ under proper assumptions. Finally, solutions to (BVP) will be deduced from an initial value problem for the matrix differential equation (4.2).

4.2.1. Connections between the spectra of L and M

To state the relation between the spectra of L and M, we need to define the sequence γ_n, $n \in \mathbb{N}^+$ recursively by

$$\gamma_1 := \frac{1}{2\lambda}, \quad \gamma_n := -\frac{\sum_{j=1}^{n-1} \gamma_j \gamma_{n-j}}{2\lambda}, \quad n \geq 2, \tag{4.4}$$

where $\lambda = \lambda^\pm := \pm\sqrt{\mu}$ is non-zero. Obviously,

$$\gamma_2 = -\frac{1}{8\lambda^3}, \; \gamma_3 = \frac{1}{16\lambda^5}, \; \gamma_4 = -\frac{5}{128\lambda^7}, \; \cdots$$

For the following lemma, recall that $h^{(j)}$ ($j = 0, 1, \cdots$) is the chain generated by h (see Definition 4.2).

Lemma 4.3. *The pair (h, μ) with $\mu \neq 0$ is an eigenpair of rank $m \geq 1$ of the differential operator L, if and only if the eigenpair (W, λ) with $\lambda = \pm\sqrt{\mu}$, $W = (h, v)^T$ and*

$$v(x_1) := \lambda\, h(x_1) + \sum_{j=1}^{m-1} \gamma_j\, h^{(j)}(x_1),$$

is an eigenpair of rank $m \geq 1$ of M.

Proof. We first consider the case $m = 1$. If (W, λ) with $W = (w, v)^\top$ is an eigenpair of rank one of M, then it is easy to conclude from $MW = \lambda W$ that $Lw = \lambda v$ and $v = \lambda w$ implying $(L - \lambda^2 I)w = 0$. Hence, $(h, \mu) = (w, \lambda^2)$ is an eigenpair of rank one of L. Similarly, it is easy to prove that, if (h, λ^2) is an eigenpair of rank one of L, then (W, λ) with $W = (h, \lambda h)^\top$ is an eigenpair of rank one of M.

Now suppose $m = 2$. If (W, λ) with $W = (w, v)^\top$, is an eigenpair of rank two of M, then $\widetilde{W} := (M - \lambda\mathbf{I})W =: (\tilde{w}, \tilde{v})^T \neq 0$ is an eigenfunction of rank one of M. This implies that $\tilde{v} = \lambda\tilde{w}$ and $(\tilde{w}, \lambda^2)$ is an eigenpair of rank one of L. From the definition of $\widetilde{W}$, it is easy to obtain that

$$-\lambda w + v = \tilde{w}, \quad Lw - \lambda v = \tilde{v}, \tag{4.5}$$

$$M^2\, W = \lambda M W + M\widetilde{W} = \lambda(\lambda W + \widetilde{W}) + M\widetilde{W} = \lambda^2 W + (M + \lambda\mathbf{I})\widetilde{W}, \tag{4.6}$$

where

$$M^2 = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}.$$

Using $\tilde{v} = \lambda \tilde{w}$, we deduce from (4.6) that

$$Lw = \lambda^2 w + (\lambda \tilde{w} + \tilde{v}) = \lambda^2 w + 2\lambda \tilde{w},$$

leading to the relations

$$\begin{aligned} &(L - \lambda^2 I)^2 w = (L - \lambda^2 I)(2\lambda \tilde{w}) = 0, \\ &\tilde{w} = \gamma_1 \, (L - \lambda^2 I) \, w \neq 0, \quad \gamma_1 := 1/(2\lambda). \end{aligned}$$

Therefore, (w, λ^2) is an eigenpair of rank two of L. From the first relation in (4.5) we obtain

$$v = \lambda w + \tilde{w} = \lambda w + \gamma_1 \, w^{(1)}, \quad w^{(j)} := (L - \lambda^2 I)^j \, w.$$

Now we treat the general case $m > 2$ by induction. Suppose the induction hypothesis

$$\begin{aligned} &\text{The pair } (W, \lambda) \text{ with } W = (w, v)^T \text{ is eigenpair of rank } m \text{ of } M \\ &\iff (w, \lambda^2) \text{ is an eigenpair of rank } m \text{ of } L \text{ and } v = \lambda w + \sum_{j=1}^{m-1} \gamma_j \, w^{(j)}, \end{aligned} \tag{4.7}$$

is fulfilled. We have to show that (4.7) holds with m replaced by $m + 1$.

⇒: Suppose that (W, λ) with $W = (w, v)^T$ is an eigenpair of rank $m + 1$ of M. Then $(\widetilde{W}, \lambda)$ with $\widetilde{W} := (M - \lambda \mathbf{I})W$ and $\widetilde{W} = (\tilde{w}, \tilde{v})^T \neq 0$ is an eigenpair of rank m of M. By induction hypotheses this implies that $(\tilde{w}, \lambda^2)$ is an eigenpair of rank m of L and

$$\tilde{v} = \lambda \tilde{w} + \sum_{j=1}^{m-1} \gamma_j \, \tilde{w}^{(j)}.$$

Combining the previous relation with (4.6) yields (cf. (4.7))

$$Lw = \lambda^2 w + (\lambda \tilde{w} + \tilde{v}) = \lambda^2 w + 2\lambda \tilde{w} + \sum_{j=1}^{m-1} \gamma_j \, \tilde{w}^{(j)},$$

from which we obtain

$$w^{(1)} := (L - \lambda^2 I)w = 2\lambda \tilde{w} + \sum_{j=1}^{m-1} \gamma_j \, \tilde{w}^{(j)}. \tag{4.8}$$

Since $(L - \lambda^2 I)^m \tilde{w} = 0$, it follows that

$$(L - \lambda^2 I)^{m+1} w = (L - \lambda^2 I)^m w^{(1)} = 2\lambda (L - \lambda^2 I)^m \tilde{w} + \sum_{j=1}^{m-1} \gamma_j \, \tilde{w}^{(m+j)} = 0$$

and

$$(L-\lambda^2 I)^m w = (L-\lambda^2 I)^{m-1} w^{(1)} = 2\lambda (L-\lambda^2 I)^{m-1}\tilde{w} \neq 0.$$

Hence, (w, λ^2) is an eigenpair of rank $m+1$ of L. To express v in terms of w, we deduce from (4.8) that

$$w^{(l)} := (L-\lambda^2 I)^l w = 2\lambda \tilde{w}^{(l-1)} + \sum_{j=1}^{m-l} \gamma_j\, \tilde{w}^{(l-1+j)}, \quad l = 1, 2, \cdots, m,$$

which form the $m \times m$ linear system of equations $\widetilde{W} = \Pi_\lambda\, \widetilde{W}'$, where $\widetilde{W} := (w^{(1)}, \cdots, w^{(m)})^T$, $\widetilde{W}' := (\tilde{w}, \tilde{w}^{(1)} \cdots, \tilde{w}^{(m-1)})^T$ and

$$\Pi_\lambda = \begin{pmatrix} 2\lambda & \gamma_1 & \gamma_2 & \cdots & \gamma_{m-1} \\ 0 & 2\lambda & \gamma_1 & \cdots & \gamma_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_1 \\ 0 & 0 & 0 & \cdots & 2\lambda \end{pmatrix}.$$

By the definition (4.4) of γ_n, the inverse of Π_λ is given by

$$\Pi_\lambda^{-1} = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_m \\ 0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{m-1} \\ 0 & 0 & \gamma_1 & \cdots & \gamma_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_1 \end{pmatrix}.$$

This implies that the first component of $\widetilde{W}'$ is given by

$$\tilde{w} = \sum_{j=1}^{m} \gamma_j\, w^{(j)}.$$

Together with the first relation in (4.5) we obtain

$$v = \lambda w + \tilde{w} = \lambda w + \sum_{j=1}^{m} \gamma_j\, w^{(j)}.$$

$\Leftarrow$: Suppose that (w, λ^2) is an eigenpair of rank $m+1$ of L and $v = \lambda w + \sum_{j=1}^m \gamma_j\, w^{(j)}$. We have to prove that $W = (w, v)^T$ is an eigenfunction of rank $m+1$ of M. It suffices to show that $\widetilde{W} = (M - \lambda \mathbf{I})W = (\tilde{w}, \tilde{v})^T$ has the rank m. By the definition of M and the expression of v from our supposition,

$$\tilde{w} = -\lambda w + v = \sum_{j=1}^{m} \gamma_j\, w^{(j)}, \tag{4.9}$$

$$\tilde{v} = Lw - \lambda v = (1-\lambda\gamma_1) w^{(1)} - \lambda \sum_{j=2}^{m} \gamma_j \, w^{(j)}. \tag{4.10}$$

Recalling the induction hypotheses, we only need to verify the relation

$$\tilde{v} = \lambda \tilde{w} + \sum_{j=1}^{m-1} \gamma_j \, \tilde{w}^{(j)}. \tag{4.11}$$

Using (4.9) and the definition (4.4) of γ_n, straightforward calculations show that

$$\begin{aligned}
\lambda \tilde{w} + \sum_{j=1}^{m-1} \gamma_j \, \tilde{w}^{(j)} &= \lambda \sum_{j=1}^{m} \gamma_j \, w^{(j)} + \sum_{j=1}^{m-1} \gamma_j \left(\sum_{l=1}^{m-j} \gamma_l \, w^{(j+l)} \right) \\
&= \lambda \sum_{j=1}^{m} \gamma_j \, w^{(j)} + \sum_{j=2}^{m} w^{(j)} \left(\sum_{l=1}^{j-1} \gamma_l \gamma_{j-l} \right) \\
&= \lambda \sum_{j=1}^{m} \gamma_j \, w^{(j)} + \sum_{j=2}^{m} w^{(j)} (-2\lambda\gamma_j) \\
&= \lambda\gamma_1 w^{(1)} - \lambda \sum_{j=2}^{m} \gamma_j \, w^{(j)}.
\end{aligned}$$

Since $2\lambda\gamma_1 = 1$ and (4.10), the previous identity confirms the relation (4.11). The proof is completed. □

The chain $W^{(j)}$ generated by W is given in Definition 4.1. As a consequence of the proof to Lemma 4.3, we obtain

Lemma 4.4.

(i) *Suppose* (h, λ^2) *is an eigenpair of rank* $m \geq 1$ *of* L. *Then the vector functions*

$$\begin{pmatrix} h^{(l)} \\ \lambda h^{(l)} + \sum_{j=1}^{m-1-l} \gamma_j \, h^{(j+l)} \end{pmatrix}, \qquad l = 0, 1, 2, \cdots, m-1,$$

are the associated eigenfunctions of rank $m - l$ *of the operator* M *corresponding to the eigenvalue* λ.

(ii) *Suppose* (W, λ) *is an eigenpair of rank* $m \geq 1$ *of operator* M. *Write* $W^{(l)} = (W_1^{(l)}, W_2^{(l)})^T$ *for* $l = 0, 1, \cdots, m-1$. *Then* $(W_1^{(l)}, \lambda^2)$ *is an eigenpair of rank* $m - l$ *of* L *and*

$$W_2^{(l)} = \lambda W_1^{(l)} + \sum_{j=1}^{m-l-1} \gamma_j \left(L - \lambda^2 \right)^j W_1^{(l)}.$$

Proof. Lemma 4.4 follows from Lemma 4.3 and the fact that $(W^{(l)}, \lambda)$, $(h^{(l)}, \lambda^2)$ are eigenpairs of rank $m-l$ corresponding to M and L, respectively. Note that, in the case of $l=0$, the assertions of Lemma 4.4 coincide with those in Lemma 4.3. □

4.2.2. Riesz property of eigenfunctions of L

By Lemma 4.3, in order to get the spectrum of M, it suffices to investigate the spectrum of the quasiperiodic differential operator L. We collect properties of the nonself-adjoint operator L in the subsequent two lemmas.

Lemma 4.5.

(i) *The spectrum $\sigma_p(L)$ of L is a discrete set of eigenvalues and the only accumulation point is infinity.*

(ii) *The geometric multiplicity of each eigenvalue $\mu \in \sigma_p(L)$ is finite, i.e.,* $\dim(\ker(L-\mu I)) < \infty$.

(iii) *The algebraic multiplicity of each eigenvalue $\mu \in \sigma_p(L)$ is finite, i.e.,* $\dim(A_L(\mu)) < \infty$, *where*

$$A_L(\mu) := \Big\{ h \in \mathcal{D} : \textit{there is an } m \in \mathbb{N} \textit{ s.t.} \quad L^j h \in \mathcal{D},\ j=1,\ldots,m-1 \textit{ and } (L-\mu I)^m h = 0 \Big\}. \tag{4.12}$$

(iv) *The eigenvalues can be denoted as $\mu_n = \mu_n(\alpha) \in \sigma_p(L)$ for index n running in $\mathbb{Z}$ and repeated according to the algebraic multiplicity.*
If $\alpha \neq 0, 1/2$, then the algebraic multiplicity of μ_n is equal to one for sufficiently large $|n|$. Choosing a suitable scaling factor for the rank-one eigenfunction h_n corresponding to μ_n, we get $h_n(0)=1$ and the asymptotics

$$\mu_n(\alpha) = (n+\alpha)^2 - \frac{k^2}{2\pi}\int_0^{2\pi} q(t)\,dt + O\left(\frac{1}{|n|}\right), \tag{4.13}$$

$$h_n(x_1) = \exp\Big(i(n+\alpha)x_1\Big) + O\left(\frac{1}{|n|}\right), \quad n \in \mathbb{Z}, \tag{4.14}$$

as $|n| \to \infty$, where the term $O(1/|n|)$ is uniform with respect to $x_1 \in [0, 2\pi]$.
If $\alpha = 0, 1/2$, then, for sufficiently large $|n|$, the algebraic multiplicity of μ_n is either one or two. The eigenvalue asymptotics (4.13) holds with $O(1/|n|)$ replaced by $O(1/|n|^{1/2})$. Instead of (4.14), the eigenfunctions of rank one admit the asymptotic expansion

$$h_n(x_1) = C_+(n)\exp[i(n+\alpha)x_1] + C_-(n)\exp[-i(n+\alpha)x_1] + O\left(\frac{1}{|n|}\right), \tag{4.15}$$

where $C_\pm(n) \in \mathbb{C}$ and $n \in \mathbb{Z}$ with $|n| \to \infty$. For normalization, in (4.15) we may suppose $h_n(0) \in \mathbb{R}$ and $|C_+(n)|^2 + |C_-(n)|^2 = 1$. Furthermore, for sufficiently large $|n|$ and for eigenvalues $\mu_n(\alpha) = \mu_{-n-2\alpha}(\alpha)$ with two linearly independent eigenfunctions of rank one, a pair of eigenfunctions h_n and $h_{-n-2\alpha}$ can be found satisfying (4.14) with n set to n and $-n-2\alpha$, respectively.

The assertions (i)-(iii) follow from the spectral theory of nonself-adjoint differential equations (see e.g., [9,15,16] and references therein). The asymptotic behavior of the spectrum of L was studied, e.g., in [39] for $\alpha \neq 0, 1/2$, in [15] for $\alpha = 0, 1/2$ and in [30] for the general case. The results in the last assertion were used in the proof of [37, Thm. 4.12] to derive uniqueness for the identification of a periodic medium, which depends only on x_2, from near-field measurement data of infinitely many incoming waves.

Obviously, one has

$$\dim(\ker(L-\mu)) \leq \dim(A_L(\mu))$$

for each $\mu \in \sigma_p(L)$, $\mu \neq 0$. The set of all eigenfunctions and associated eigenfunctions of $\mu \in \sigma_p(L)$ form the eigenspace corresponding μ, which is a closed linear subspace of $L^2(0, 2\pi)$ with dimension equal to the algebraic multiplicity of μ. For $q = 0$ and $n \in \mathbb{Z}$, we have $\mu_n = (n+\alpha)^2$ and all associated eigenfunctions $h_n(x_1) = \exp\big(i(n+\alpha)x_1\big)$ are of rank one. For $q \neq 0$, the eigenvalues as well as the eigenfunctions and associated eigenfunctions are obtained by perturbation arguments. Therefore, we have the same general indices $n \in \mathbb{Z}$ for the set of all eigenfunctions and associated eigenfunctions. So this covers the case of associated eigenfunctions of rank greater than one. Indeed, in this case the values μ_n might coincide for several $n \in \mathbb{Z}$ and the corresponding h_n span the space of all eigenfunctions and associated eigenfunctions.

Since the α-quasiperiodic boundary conditions are non-degenerate, we infer from [30, Thm. 1.3.1], [15, Thm. 2.1] and [39, Thm. 3] that

Lemma 4.6. *The system of eigenfunctions and associated eigenfunctions $h_n$of the α-quasiperiodic operator L is complete over* $L^2(0, 2\pi)$. *Further, they form a Riesz basis of* $L^2(0, 2\pi)$ *if* $\alpha \neq 0, 1/2$.

Let us comment on the choice of eigenfunctions for a basis. Note that, for $\alpha \neq 0, 1/2$, each eigenvalue μ_n with sufficiently large $|n|$ has an eigenfunction of rank one, which is unique by the normalization $h_n(0) = 1$. A basis transform for the general eigenfunctions with n in a finite set does not change the Riesz property. For $\alpha = 0, 1/2$, the eigenvalues of multiplicity two have a non-unique basis. If the two eigenfunctions are both of rank one, then the basis can be fixed by $h_n(0) = 1$ and (4.14) without changing the Riesz property. However, if there is a generalized eigenfunction of rank two, then the Riesz property might depend on a good choice of generalized eigenfunctions for the basis. In particular, it might be necessary to choose two eigenfunctions of rank two for some of the eigenvalues in order to form a Riesz basis. Choosing a chain of generalized eigenfunctions might lead to a system without Riesz property. We suppose that the system of generalized eigenfunctions h_n is chosen such that the Riesz property is fulfilled whenever this is possible. Moreover, we assume a special choice of rank-two eigenfunctions. For this purpose we define

Definition 4.7. The set I_d is defined as the set of indices n such that $\mu_n = \mu_{-n-2\alpha}$ has at least one rank-two eigenfunction h_n or $h_{-n-2\alpha}$ in the Riesz system.

Then, for $n \in I_d$,

$$\begin{aligned} \big([-\partial^2 - k^2 qI] - \mu_n\big)h_n &= c_{n,1,1}h_n + c_{n,1,2}h_{-n-2\alpha}, \\ \big([-\partial^2 - k^2 qI] - \mu_n\big)h_{-n-2\alpha} &= c_{n,2,1}h_n + c_{n,2,2}h_{-n-2\alpha}. \end{aligned} \tag{4.16}$$

For a linear combination $f_n h_n + f_{-n-2\alpha} h_{-n-2\alpha}$ with $f_n, f_{-n-2\alpha} \in \mathbb{C}$, we get

$$\left\| \partial^2 \left(f_n h_n + f_{-n-2\alpha} h_{-n-2\alpha} \right) \right\|^2 \sim \left\langle B_n^* B_n (f_n, f_{-n-2\alpha})^\top, (f_n, f_{-n-2\alpha})^\top \right\rangle,$$
$$B_n := \begin{pmatrix} \mu_n + c_{n,1,1} & c_{n,2,1} \\ c_{n,1,2} & \mu_n + c_{n,2,2} \end{pmatrix}.$$

By the eigenvalue decomposition of self-adjoint matrices there exists a unitary matrix U_n and non-negative eigenvalues $\kappa_n, \kappa_{-n-2\alpha}$ such that

$$B_n^* B_n = U_n^* \operatorname{diag}(\kappa_n, \kappa_{-n-2\alpha}) U_n. \tag{4.17}$$

In other words, applying a basis transform for the basis functions h_n and $h_{-n-2\alpha}$, we may suppose $U_n = I$ and arrive at

$$\left\| \partial^2 \left(f_n h_n + f_{-n-2\alpha} h_{-n-2\alpha} \right) \right\|^2 \sim \kappa_n |f_n|^2 + \kappa_{-n-2\alpha} |f_{-n-2\alpha}|^2. \tag{4.18}$$

This normalization of pairs of basis functions for $\alpha = 0, 1/2$ will always be supposed in the following. If $\alpha \neq 0, 1/2$, then we set $I_d = \emptyset$, since, for large $|n|$, all eigenvalues μ_n have algebraic multiplicity one.

The adjoint operator of L over the quasiperiodic functions is the operator L^* over quasiperiodic functions, which is defined as L in (4.1) but with q replaced by the complex conjugate function $\overline{q}$. Since the eigenfunctions and the associated eigenfunctions of L^* corresponding to $\overline{\mu_n}$ are L^2 orthogonal to the eigenfunctions and associated eigenfunctions of L corresponding to μ_m for $\mu_m \neq \mu_n$ (cf. the proof of [39, Thm. 3]), we conclude that there exists a dual system h_n^*, $n \in \mathbb{Z}$ such that $\langle h_m^*, h_n \rangle = \delta_{m,n}$ and $\langle h_m^*, h_n \rangle = \delta_{m,n}$. The existence of a complete dual system implies that the system h_n, $n \in \mathbb{Z}$ is total and minimal. Of course, the scaling for the dual system is different than that in Lemma 4.5, (iv). In particular, if the algebraic multiplicity of an eigenvalue is greater than one, then the scaling is difficult to estimate and the Riesz property might get lost.

If $\alpha = 0, 1/2$, then the α-quasiperiodic boundary conditions reduce to the periodic boundary conditions and the antiperiodic boundary conditions $h(0) = -h(2\pi)$, $h'(0) = -h'(2\pi)$, respectively. Unfortunately, the modified asymptotics (4.13) does not exclude the identity $\mu_n(\alpha) = \mu_{-n+2\alpha}(\alpha)$ for large $|n|$, which might lead to troubles in estimating the norms of the dual basis. We refer to [15, Thm. 1.2, Cor. 1.5] for necessary and sufficient conditions, under which the eigenfunctions form a Riesz or Schauder basis over $L^2(0, 2\pi)$ in the case of $\alpha = 0, 1/2$.

For general α but real-valued q, the operator L over quasiperiodic functions is self-adjoint and the system h_n, $n \in \mathbb{Z}$ forms an orthogonal basis in the Hilbert space L^2. In this paper we suppose that either $\alpha \neq 0, 1/2$, or q is real-valued, or the conditions in [15, Thm. 1.2, Cor. 1.5] hold for $\alpha = 0, 1/2$, so that the h_n, $n \in \mathbb{Z}$ always form a Riesz basis. Note that, for the main result in Theorem 5.7, the Riesz basis assumption can be replaced by assuming a subexponential bound for the norms of the dual basis. However, this leads to more involved definitions and proofs, since the convergence of an expansion with respect to a Riesz basis is to be replaced by density arguments for finite linear combinations of the h_n, $n \in \mathbb{Z}$. With the Riesz basis assumption, for each α we obtain the following equivalence of the Sobolev norms with weighted ℓ^2 norms of the coefficients with respect to the Riesz basis h_n, $n \in \mathbb{Z}$.

Lemma 4.8. *Suppose h_n, $n \in \mathbb{Z}$ is a Riesz basis in $L^2(0, 2\pi)$. For each s fixed with $-2 \le s \le 2$, there exists a constant $c_s > 0$ such that, for all sequences $f_n \in \mathbb{C}$,*

$$\frac{1}{c_s}\Big\|\sum_{n\in\mathbb{Z}} f_n h_n\Big\|^2_{H^s_\alpha(0,2\pi)} \le \sum_{n\in\mathbb{Z}\setminus I_d}(1+|n|)^{2s}|f_n|^2 + \sum_{n\in I_d}(1+\kappa_n)^s|f_n|^2$$
$$\le c_s\Big\|\sum_{n\in\mathbb{Z}} f_n h_n\Big\|^2_{H^s_\alpha(0,2\pi)},$$

where κ_n and I_d are given by (4.7) and Definition 4.17, respectively. Moreover, we have $\kappa_n \le O(|n|^4)$ as $|n| \to \infty$.

Proof. For $s = 0$ the norm equivalence is a well-known fact for any kind of Riesz basis. If $s = 2$ and all eigenfunctions with eigenvalue $\mu_n \ge n_0$ are of rank one, then

$$\Big\|\sum_{n\in\mathbb{Z}} f_n h_n\Big\|^2_{H^2_\alpha(0,2\pi)} \sim \Big\|\sum_{n\in\mathbb{Z}} f_n\, h_n''\Big\|^2_{L^2_\alpha(0,2\pi)} + \Big\|\sum_{n\in\mathbb{Z}} f_n h_n\Big\|^2_{L^2_\alpha(0,2\pi)} \tag{4.19}$$
$$\sim \Big\|\sum_{n\in\mathbb{Z}:|n|\ge n_0} f_n(\mu_n + k^2 q)h_n\Big\|^2_{L^2_\alpha(0,2\pi)} + \sum_{n\in\mathbb{Z}}|f_n|^2.$$

Using $q \in L^\infty(0, 2\pi)$ and the fact that $\mu_n \sim |n|^2$ for $n \to \pm\infty$ (see Lemma 4.5, (iv)) we continue

$$\Big\|\sum_{n\in\mathbb{Z}} f_n h_n\Big\|^2_{H^2_\alpha(0,2\pi)} \sim \Big\|\sum_{n\in\mathbb{Z}:|n|\ge n_0} (\mu_n f_n)h_n\Big\|^2_{L^2_\alpha(0,2\pi)} + \sum_{n\in\mathbb{Z}}|f_n|^2$$
$$\sim \sum_{n\in\mathbb{Z}}|\mu_n|^2|f_n|^2 + \sum_{n\in\mathbb{Z}}|f_n|^2$$
$$\sim \sum_{n\in\mathbb{Z}}(1+|n|)^4|f_n|^2.$$

Hence, the assertion holds for $s = 2$, and the norm of the dual space $H^{-2}_\alpha(0, 2\pi)$ is equivalent to dual of the weighted ℓ^2 space, i.e., the assertion is true for $s = -2$. By interpolating the spaces, we obtain the assertion for any s with $-2 \le s \le 2$.

The proof in the general case follows analogously, if we apply $h_n'' = (\mu_n + k^2 q)h_n + g_n$ instead of $h_n'' = (\mu_n + k^2 q)h_n$ to (4.19) and if we use (4.18). It remains to show the estimate of the κ_n. If $n \in I_d$, then we get (4.16). We denote the rank-one eigenfunction on the right-hand side of (4.16) by g_n. Fixing a suitable $c_0 > 0$, the operator $[(-\partial^2 - k^2 qI) + c_0 I]$ is invertible and its inverse is the compact resolvent operator $B := [(-\partial^2 - k^2 qI) + c_0 I]^{-1}$. Hence, the property $(-\partial^2 - k^2 qI)g_n = \mu_n g_n$ of the rank-one eigenfunction g_n leads us to

$$[(-\partial^2 - k^2 qI) + c_0 I]h_n - (\mu_n + c_0)h_n = g_n,$$
$$(\mu_n + c_0)^{-1}h_n - Bh_n = (\mu_n + c_0)^{-2}g_n,$$
$$g_n = (\mu_n + c_0)h_n - (\mu_n + c_0)^2 Bh_n.$$

Here $\|(\mu_n + c_0)h_n\| = O(|n|^2)$, and B is a bounded operator in L^2. Thus $\|g_n\| = O(|n|^4)$ such that $c_{n,1,j} = O(|n|^4)$, $j = 1, 2$. Similarly, $c_{n,2,j} = O(|n|^4)$, $j = 1, 2$, and the non-negative singular value κ_n is at most $O(|n|^4)$. □

4.2.3. Solutions to the BVP (2.5)

By Lemma 4.6, the set of eigenfunctions and associated eigenfunctions of L is complete over $L^2(0, 2\pi)$ for any $\alpha \in [0, 1)$. To consider eigenfunctions of higher ranks, we denote by $(h_{n,m}, \mu_n)$ with $h_{n,m} \in A_L(\mu_n)$ an eigenpair of rank $m \geq 1$ of L. However, we should always keep in mind that the system $(h_{n,m}, \mu_n)$ coincides with the previously used notation (h_n, μ_n). By Lemma 4.3 we may construct eigenpairs $(W^{\pm}_{n,m}, \lambda^{\pm}_n)$ of rank $m \geq 1$ of M as follows:

$$\lambda^{\pm}_n = \pm\sqrt{\mu_n}, \quad W^{\pm}_{n,m}(x_1) = \begin{pmatrix} h_{n,m}(x_1) \\ \lambda^{\pm}_n\, h_{n,m}(x_1) + \sum_{j=1}^{m-1} \gamma^{\pm}_{j,n}\, h^{(j)}_{n,m}(x_1) \end{pmatrix} \in A_M(\lambda^{\pm}_n), \tag{4.20}$$

where the $\gamma^{\pm}_{j,n}$ are defined the same way as γ_j with λ replaced by $\lambda^{\pm}_n$ (see (4.4)). Here, the functions $h^{(j)}_{n,m} = (L - \mu_n I)^j h_{n,m}$ represent the chain generated by $h_{n,m}$ and the set $A_M(\lambda)$ denotes the eigenspace of the operator M corresponding to the eigenvalue λ, that is (cf. (4.12)),

$$A_M(\lambda) := \Big\{ g \in \mathcal{D}^2 : \text{there is an } m \in \mathbb{N} \text{ s.t.} \\ M^j g \in \mathcal{D}^2,\ j = 1, \ldots, m-1 \text{ and } (M - \lambda \mathbf{I})^m\, g = 0 \Big\}.$$

As will be seen later, we shall switch between the indices + and − to define upward and downward radiating wave modes for $x_2 \geq b$ and $x_2 \leq b$, respectively.

Lemma 4.9. *Suppose (g, λ) with $g = (g_1, g_2)^\top \in A_M(\lambda)$ is an eigenpair of rank $m \geq 1$ of M. Then the unique solution $W(x_1, x_2) = (u(x_1, x_2), v(x_1, x_2))^\top$ to the quasiperiodic initial boundary value problem*

$$\partial_2 W = M\, W, \quad W(\cdot, b) = g, \tag{4.21}$$

is given by

$$W(x_1, x_2) = e^{\lambda(x_2 - b)} \sum_{n=0}^{m-1} \frac{g^{(n)}(x_1)\,(x_2 - b)^n}{n\,!},$$

where $\{g^{(n)} : n = 1, \cdots, m\}$ denotes the chain generated by g as defined for generator h in Definition 4.2.

Proof. Without loss of generality we suppose that b=0. Obviously, $W(x_1, x_2) := \exp(M x_2) g(x_1)$ is the unique solution to (4.21). For $m = 1$, we have $(M - \lambda \boldsymbol{I})g = 0$, implying that $M^j g = \lambda^j g$ for any $j \in \mathbb{N}$. Hence, by the definition of the exponential function of a matrix we obtain

$$W(x_1,x_2)=\exp(Mx_2)g(x_1)=\sum_{j=0}^{\infty}\frac{x_2^j}{j!}M^j g=\sum_{j=0}^{\infty}\frac{x_2^j\lambda^j}{j!}g=e^{\lambda x_2}g.$$

Next we will verify the lemma in the general case of $m\geq 1$. From the definition of $g^{(n)}$, using an induction argument we see

$$M^j\,g=\sum_{n=0}^{\min\{j,m-1\}}\lambda^{j-n}\,g^{(n)}\begin{pmatrix}j\\n\end{pmatrix},\quad\begin{pmatrix}j\\n\end{pmatrix}:=\frac{j\,!}{(j-n)\,!\,n\,!}\,. \tag{4.22}$$

Note that in deriving (4.22), we have used the relation $Mg^{(n)}=\lambda g^{(n)}+g^{(n+1)}$. We split the function $e^{Mx_2}g$ into the sum of

$$\exp(Mx_2)g(x_1)=\sum_{j=0}^{m-1}\frac{x_2^j}{j!}M^j g+\sum_{j=m}^{\infty}\frac{x_2^j}{j!}M^j g\,. \tag{4.23}$$

The first sum on the right-hand side of the previous identity can be rewritten using (4.22) as

$$\begin{aligned}\sum_{j=0}^{m-1}\frac{x_2^j}{j!}M^j g&=\sum_{j=0}^{m-1}\frac{x_2^j}{j!}\sum_{n=0}^{j}\lambda^{j-n}\,g^{(n)}\begin{pmatrix}j\\n\end{pmatrix}\quad=\sum_{j=0}^{m-1}x_2^j\sum_{n=0}^{j}\frac{\lambda^{j-n}}{(j-n)\,!\,n\,!}g^{(n)}\\&=\sum_{n=0}^{m-1}\frac{x_2^n}{n\,!}g^{(n)}\sum_{j=n}^{m-1}\frac{x_2^{j-n}\,\lambda^{j-n}}{(j-n)\,!},\end{aligned}$$

where the summation over the indices j and m has been interchanged in the last step. Analogously,

$$\sum_{j=m}^{\infty}\frac{x_2^j}{j!}M^j g=\sum_{n=1}^{m-1}\frac{x_2^n}{n\,!}g^{(n)}\sum_{j=m}^{\infty}\frac{x_2^{j-n}\,\lambda^{j-n}}{(j-n)\,!}.$$

The previous two identities together with (4.23) imply

$$\exp(Mx_2)g=\sum_{n=0}^{m-1}\frac{x_2^n}{n\,!}g^{(n)}\left(\sum_{j=0}^{\infty}\frac{x_2^j\,\lambda^j}{j\,!}\right)=e^{\lambda x_2}\sum_{n=0}^{m-1}\frac{x_2^n}{n\,!}g^{(n)}.\quad\square$$

Theorem 4.10. *Suppose $(h_{n,m},\mu_n)$ with $h_{n,m}\in A_L(\mu_n)$ is an eigenpair of rank $m\geq 1$ of L and define $\lambda_n^{\pm}$ and $W_{n,m}^{\pm}$ as in (4.20). Consider the boundary value problem for α-quasiperiodic solutions u:*

$$\Delta u+k^2qu=0\quad in\quad\mathbb{R}^2,\qquad u=h_{n,m}\quad on\quad\Gamma_b, \tag{4.24}$$

(i) The general solution $u=u_{n,m}\in H^2_{loc}(\mathbb{R}^2)$ can be represented by $u_{n,m}=C^+u^+_{n,m}+C^-u^-_{n,m}$, where $C^{\pm}\in\mathbb{C},\ C^++C^-=1$, and

$$u^{\pm}_{n,m}(x_1,x_2) = e^{\lambda^{\pm}_n(x_2-b)} \sum_{j=0}^{m-1} (W^{\pm}_{n,m})^{(j)}_1(x_1)\,\frac{(x_2-b)^j}{j\,!}. \tag{4.25}$$

Here $(W^{\pm}_{n,m})^{(j)}_1$ *denotes the first component of the chain* $(W^{\pm}_{n,m})^{(j)}$ *generated by* $W^{\pm}_{n,m}$. *Furthermore, for* $0 \le j \le m-1$, *the associated eigenfunction* $(W^{\pm}_{n,m})^{(j)}$ *of the operator* M *with the corresponding eigenvalue* $\lambda^{\pm}_n$ *is of rank* $m-j$ *and can be represented as*

$$(W^{\pm}_{n,m})^{(j)}_1 = \sum_{l=0}^{m-1} A^{(j)}_l\, h^{(l)}_{n,m}, \quad (W^{\pm}_{n,m})^{(j)}_2 = \sum_{l=0}^{m-1} B^{(j)}_l\, h^{(l)}_{n,m}, \quad 0 \le j \le m-1, \tag{4.26}$$

with the coefficients $A^{(j)}_l = A^{\pm,(j)}_l$, $0 \le l \le m-1$ *and* $B^{(j)}_l = B^{\pm,(j)}_l$, $0 \le l \le m-1$ *given by the recursion*

$$A^{(0)}_0 := 1,\ B^{(0)}_0 := \lambda^{\pm}_n,\ A^{(0)}_l := 0,\ B^{(0)}_l := \gamma^{\pm}_{l,n}, \qquad 0 < l \le m-1, \tag{4.27}$$

$$A^{(j+1)}_l = -\lambda^{\pm}_n A^{(j)}_l + B^{(j)}_l,\ B^{(j+1)}_l = A^{(j)}_{l-1} + \mu_n A^{(j)}_l - \lambda^{\pm}_n B^{(j)}_l,\ 0 \le j \le m-1, \tag{4.28}$$

where $\mu_n = [\lambda^{\pm}_n]^2$ *and* $A^{(j)}_{-1} := 0$.

(ii) It holds that

$$\partial_2 u^{\pm}_{n,m}(x_1,b) = \lambda^{\pm}_n h_{n,m}(x_1) + \sum_{j=1}^{m-1} \gamma^{\pm}_{j,n}\, h^{(j)}_{n,m}(x_1).$$

Proof. Suppose $\lambda^{\pm}_n$ and $W^{\pm}_{n,m}$ are defined by (4.20). By Lemma 4.3, the eigenpairs $(W^{\pm}_{n,m}, \lambda^{\pm}_n)$ of M are of rank m. Hence, the $u^{\pm}_{n,m}$ are solutions of the α-quasiperiodic boundary value problem (4.24) if and only if $W^{\pm} = (u^{\pm}_{n,m}, \partial_2 u^{\pm}_{n,m})$ satisfy the α-quasiperiodic ODE systems

$$\partial_2 W^{\pm} = M W^{\pm} \quad \text{in} \quad \mathbb{R}^2, \qquad\qquad W^{\pm} = W^{\pm}_{n,m} \quad \text{on} \quad \Gamma_b.$$

By Lemma 4.9, we get the solutions

$$W^{\pm}(x_1,x_2) = e^{\lambda^{\pm}_n(x_2-b)} \sum_{j=0}^{m-1} (W^{\pm}_{n,m})^{(j)}(x_1)\frac{(x_2-b)^j}{j\,!}. \tag{4.29}$$

Recall from (4.20) that

$$(W^{\pm}_{n,m})^{(0)}_1 = (W^{\pm}_{n,m})_1 = h_{n,m},$$
$$(W^{\pm}_{n,m})^{(0)}_2 = \lambda^{\pm}_n\, h_{n,m} + \sum_{j=1}^{m-1} \gamma^{\pm}_{j,n}\, h^{(j)}_{n,m}.$$

The expression of $u^{\pm}_{n,m}$ follows from the first component of (4.29), and consequently, $\partial_2 u^{\pm}_{n,m}|_{\Gamma_b}$ coincides with the second component of $W^{\pm}_{n,m}|_{\Gamma_b}$. Finally, the initial condition (4.27) follows from (4.20) and the recursion (4.28) for the coefficients in (4.26) from

$$(M-\lambda_n^\pm \mathbf{I})=\begin{pmatrix} -\lambda_n^\pm I & I \\ (L-\mu_n I)+\mu_n I & -\lambda_n^\pm I \end{pmatrix}. \quad \square$$

As a consequence of Theorem 4.10, we present the solutions for eigenvalues of rank two.

Corollary 4.11. *Suppose (h,λ^2) with $h\in A_L(\lambda^2)$ is an eigenpair of L of rank two. Then the general solution $u\in H^2_{loc}(\mathbb{R}^2)$ of the boundary value problem (4.24) can be represented by $u=C^+u^++C^-u^-$, where $C^\pm\in\mathbb{C}$, $C^++C^-=1$, and*

$$u^\pm(x_1,x_2)=e^{\pm\lambda(x_2-b)}\left[h(x_1)\pm\frac{1}{2\lambda}(x_2-b)\,h^{(1)}(x_1)\right],\quad x\in\mathbb{R}^2,$$

where $h^{(1)}=(L-\lambda^2 I)h\neq 0$. In particular, we have

$$\partial_2 u^\pm(x_1,b)=\pm\lambda h(x_1)\pm\frac{1}{2\lambda}h^{(1)}(x_1)\qquad \textit{for}\quad x_2=b.$$

Proof. The assertion follows from Theorem 4.10 with the following replacement

$$m=2,\ \lambda_n^\pm=\pm\lambda,\ \gamma_{1,n}^\pm=\frac{1}{2\lambda_n^\pm}=\pm\frac{1}{2\lambda},\quad u_{n,2}^\pm=u^\pm,\ h_{n,2}=h.\quad \square$$

Remark 4.12. Since $\lambda_n^\pm=\pm\sqrt{\mu_n}\neq 0$, the solutions $u_{n,m}^+$ are upward outgoing, whereas $u_{n,m}^-$ are downward outgoing. They constitute a basis of the wave modes to define upward and downward radiating conditions (cf. Subsect. 4.4 below).

4.3. The eigenvalue zero

In this subsection we suppose that $\mu=0$ is an eigenvalue of L with the eigenfunction h. If $(h,0)^\top$ is an eigenpair of rank one, by Theorem 4.10 the solution u to the quasiperiodic boundary value problem (4.24) takes the form

$$u(x)=h(x_1),\quad x\in\mathbb{R}^2, \tag{4.30}$$

implying that $\partial_2 u(x_1,x_2)=0$ for any (x_1,x_2). For higher ranks $m\geq 2$, however, Theorem 4.10 is not meaningful because the coefficients γ_j, $j\geq 1$ (cf. (4.4)) are not well defined for eigenvalue zero.

Lemma 4.13. *Suppose $\lambda=0$ is an eigenvalue for M of rank $2m-1$ or $2m$ with $m\geq 1$. Then the corresponding eigenspace of rank $2m-1$ consists of vector functions of the form $(u_m,v_{m-1})^T$, while the eigenspace of rank $2m$ consists of functions of the form $(u_m,v_m)^T$. Here the u_m,v_m and v_{m-1} ($v_0\equiv 0$) are eigenfunctions of L with respect to the eigenvalue $\mu=0$ of rank m and $m-1$, respectively.*

Proof. Denote by $W=(u,v)^T$ the eigenfunction of M that corresponds to the eigenvalue $\lambda=0$. It is easy to see

$$MW = \begin{pmatrix} 0 & 1 \\ L & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ Lu \end{pmatrix}. \tag{4.31}$$

Hence, $(W, 0)$ is an eigenpair of rank one if and only if $v = 0$ and $Lu = 0$, that is $W = (u, 0)$, where the eigenvector u of L corresponding to the eigenvalue zero is of rank one. Analogously, $(W, 0)$ is an eigenpair of rank two if and only if $(v, Lu)^T$ is an eigenfunction of rank one, which implies that v is of rank one and $Lu = 0$, that is both v and u are of rank one. This proves Lemma 4.13 in the cases $m = 1$ and $m = 2$. The general case $m \geq 3$ can be proved easily via induction and using (4.31). □

Theorem 4.14. *Suppose $(h_{0,m}, 0)$, $h_{0,m} \in A_L(0)$ is an eigenpair of operator L with rank $m \geq 1$. Then the general solution $u \in H^2_{loc}(\mathbb{R}^2)$ to the quasiperiodic boundary value problem (4.24) takes the form $u = C^+ u_m^+ + C^- u_m^-$, where $C^\pm \in \mathbb{C}$, $C^+ + C^- = 1$, and*

$$u_m^\pm(x_1, x_2) = \sum_{j=0}^{2m-1} w_m^{j,\pm}(x_1)\,(x_2 - b)^j / j\,!, \qquad x_2 \in \mathbb{R},$$

where, for $n = 0, 1, \cdots, m-1$,

$$w_m^{2n,\pm}(x_1) = h_{0,m}^{(n)}(x_1), \quad w_m^{2n+1,+}(x_1) = v_m^{(n)}(x_1), \quad w_m^{2n+1,-}(x_1) = v_{m-1}^{(n)}(x_1). \tag{4.32}$$

Here v_m, v_{m-1} ($v_0 \equiv 0$) are arbitrary eigenfunctions of L of rank m and $m-1$, respectively, and $v_m^{(n)} := L^n v_m$ denotes the chain generated by v_m corresponding to operator L and eigenvalue zero. In particular, it holds that

$$\partial_2 u_m^+(x_1, b) = v_m(x_1), \quad \partial_2 u_m^-(x_1, b) = v_{m-1}(x_1),$$

Proof. By Lemma 4.13, the vector functions $W_m^+ := (h_m, v_m)^T$, $W_m^- := (h_m, v_{m-1})^T$ are of rank $2m$ and $2m-1$, respectively. Now, consider the quasiperiodic boundary value problems

$$\partial_2 W^\pm = M W^\pm, \qquad W^\pm(\cdot, b) = W_m^\pm,$$

where $W^\pm = (u_m^\pm, \partial_2 u_m^\pm)^T$. By Lemma 4.9, we have the solution

$$W^\pm(x) = \sum_{j=0}^{2m-1} (W_m^\pm)^{(j)}(x_1)\,(x_2 - b)^j / j\,!,$$

where $(W_m^\pm)^{(j)} = M^j W_m^\pm$ denotes the chain generated by $W_m^\pm$. By the definition of M, we get

$$(W_m^+)^{(2n)} = \begin{pmatrix} h_m^{(n)} \\ v_m^{(n)} \end{pmatrix}, \quad (W_m^+)^{(2n+1)} = \begin{pmatrix} v_m^{(n)} \\ h_m^{(n+1)} \end{pmatrix}, \quad n = 0, 1, \cdots, m-1.$$

The first component of $(W_m^+)^{(j)}$, $j = 0, 1, \cdots, 2m-1$ gives the definition of $w_m^{j,+}$ in (4.32). Analogously, we can get

$$(W_m^-)^{(2n)} = \begin{pmatrix} h_m^{(n)} \\ v_{m-1}^{(n)} \end{pmatrix}, \quad (W_m^-)^{(2n+1)} = \begin{pmatrix} v_{m-1}^{(n)} \\ h_m^{(n+1)} \end{pmatrix}, \quad n = 0, 1, \cdots, m-1,$$

which imply the expressions of $w_m^{j,-}$. The representation of $\partial_2 u_m^\pm$ on $x_2 = b$ follows from the expression of $u_m^\pm$ and definition of $w_m^{1,\pm}$. $\square$

In the case of $m = 1$, we have

$$u_1^+(x) = h_1(x_1) + (x_2 - b)v_1(x_1), \quad u_1^-(x_1) = h_1(x_1).$$

For $m \geq 1$, the functions $u_m^\pm(x_1, x_2)$ are polynomials with respect to x_2 of order $2m-1$ and $2m-2$, respectively. Since u_1^+ and $u_m^\pm$ ($m \geq 2$) are unbounded as $x_2 \to \pm\infty$, these wave modes are physically not meaningful. Hence, in this paper we make the assumption that the rank of $\mu = 0$ of L is one and the corresponding eigenfunction is given by $u = u_1^- = h_1(x_1)$, which coincides with the solution obtained by Theorem 4.10 by formally setting $\mu_n = 0$ and $m = 1$ (cf. (4.30)). Note that for complex-valued periodic potential $q \in L^\infty(\mathbb{R})$, one cannot exclude, in general, that zero has an associated eigenfunction of rank $m \geq 2$.

4.4. Upward and downward radiation conditions

Suppose the operator L in (4.1) is defined with a function $q \in L^\infty(\mathbb{R})$. We introduce the following assumption on L.

Definition 4.15. We shall say that Assumption RC(q) is fulfilled if the system of eigenfunctions corresponding to L (cf. Lemma 4.6) forms a Riesz basis and if either there is no eigenvalue zero of L or any eigenfunction u of eigenvalue zero is of rank one, i.e., $L^2 u = 0$ implies $Lu = 0$.

We suppose the space is filled with material, the refractive index $\tilde{q}(x)$ of which is equal to $q^+(x_1)$ and to $q^-(x_1)$ in an upper and lower half space, respectively. Denote the operator L of (4.1) with $q = q^\pm$ by $L^\pm$. In this and the following sections we shall assume the Assumptions RC($q^\pm$). For $q = q^\pm$ and $L = L^\pm$, the Riesz basis $\{h_n : n \in \mathbb{Z}\}$ can be denoted by $\{h_{n,m} : \tilde{\mu}_n \in \sigma_p(L), h_{n,m} \in A_L^F(\tilde{\mu}_n)\}$ with a finite subset $A_L^F(\tilde{\mu}_n) \subset A_L(\tilde{\mu}_n)$ (cf. (4.12)). Whereas the eigenvalues μ_n, $n \in \mathbb{Z}$ in Lemma 4.5, point (iv) need not to be different for different indices n, the eigenvalues $\tilde{\mu}_n$, $n \in \mathbb{N}$ in the new notation satisfy $\tilde{\mu}_i \neq \tilde{\mu}_j$, $i, j = 1, \cdots$ and $\operatorname{Re} \tilde{\mu}_1 \leq \operatorname{Re} \tilde{\mu}_2 \leq \operatorname{Re} \tilde{\mu}_3 \leq \cdots$. Setting $I := \{(n, m) : n \in \mathbb{N},\, m \in A_L^F(\mu_n)\}$, we can even write the system as $\{h_{n,m} : (n, m) \in I\}$. The subscript $m \geq 1$ indicates the rank m of eigenfunction $h_{n,m}$, and the corresponding set of eigenpairs is $\{(h_{n,m}, \tilde{\mu}_n) : (n, m) \in I\}$. To simplify notation we even write μ_n for the new $\tilde{\mu}_n$. Furthermore, suppose $u_{n,m}^\pm$ is given by (4.25) and let $\lambda_n^\pm$ and $W_{n,m}^\pm$ be defined as in (4.20). Set

$$\hat{\lambda}_n := \begin{cases} \sqrt{\mu_n} & \text{if} \quad \operatorname{Re}\sqrt{\mu_n} < 0 \quad \text{or} \quad \operatorname{Re}\sqrt{\mu_n} = 0,\ \operatorname{Im}\sqrt{\mu_n} \geq 0, \\ -\sqrt{\mu_n} & \text{otherwise.} \end{cases} \tag{4.33}$$

It is clear that we always have either $\operatorname{Re}(\hat{\lambda}_n) < 0$ or $\operatorname{Re}(\hat{\lambda}_n) = 0$ and $\operatorname{Im}(\hat{\lambda}_n) \geq 0$. Similarly, define

$$\widehat{W}_{n,m} := \begin{cases} W^{+}_{n,m} & \text{if } \operatorname{Re}\sqrt{\mu_n} < 0 \quad \text{or} \quad \operatorname{Re}\sqrt{\mu_n} = 0,\ \operatorname{Im}\sqrt{\mu_n} \geq 0, \\ W^{-}_{n,m} & \text{otherwise.} \end{cases}$$

Note that, for $\hat{\lambda}_n = 0$, we have $m = 1$ and $\widehat{W}_{n,m} = \widehat{W}_{n,1} = (h_{n,1}, 0)^T$, where $h_{n,1} = h_1$ denotes the eigenfunction of rank one that corresponds to the eigenvalue zero and operator L.

Definition 4.16. An upward (resp. downward) radiating mode $u^{(U)}_{n,m}$ (resp. $u^{(D)}_{n,m}$) is defined as

$$u^{(U)}_{n,m} = e^{\hat{\lambda}_n (x_2 - b)} \sum_{j=0}^{m-1} (\widehat{W}_{n,m})^{(j)}_1 (x_1) \frac{(x_2 - b)^j}{j\,!}, \quad x_2 \geq b,$$

$$u^{(D)}_{n,m} = e^{-\hat{\lambda}_n (x_2 - b)} \sum_{j=0}^{m-1} (\widehat{W}_{n,m})^{(j)}_1 (x_1) \frac{(x_2 - b)^j}{j\,!}, \quad x_2 \leq b.$$

We shall call the modes $u^{(U)}_{n,m}$ and $u^{(D)}_{n,m}$ propagating wave mode if $\operatorname{Re}\hat{\lambda}_n = 0$, i.e., if it is not decaying exponentially for $x_2 \to \infty$ and $x_2 \to -\infty$, respectively.

Remark 4.17. Each upward and downward radiating mode belongs to $H^2_{loc}(\mathbb{R}^2)$. For $\alpha \neq 0, 1/2$ and for $|n|$ sufficiently large, by Lemma 4.5 (iv) the eigenpair (h_n, μ_n) of L has the rank one. Together with Theorem 4.10, this implies that

$$u^{(U)}_{n,m} = u^{(U)}_n = e^{\hat{\lambda}_n (x_2 - b)} h_n, \quad u^{(D)}_{n,m} = u^{(D)}_n = e^{-\hat{\lambda}_n (x_2 - b)} h_n.$$

Independent on whether the rank is one or two, for large $|n|$ the function $u^{(U)}_n$ (resp. $u^{(D)}_n$) decays exponentially as $x_2 \to +\infty$ (resp. $x_2 \to -\infty$), due to the definition of $\hat{\lambda}_n$ and the asymptotics of $\hat{\lambda}_n$ shown in Lemma 4.5 (iv).

Definition 4.18. The α-quasiperiodic function $u \in H^1_{loc}(\Omega^+_b)$ (resp. $u \in H^1_{loc}(\Omega^+_b)$) is called an upward (resp. downward) radiating solution if u is a linear combination of the upward (resp. downward) radiating modes, that is,

$$u(x) = \sum_{(n,m)\in I} C^{+}_{n,m}\, u^{(U)}_{n,m}(x),$$

$$\text{(resp.)} \quad u(x) = \sum_{(n,m)\in I} C^{-}_{n,m}\, u^{(D)}_{n,m}(x),$$

for some sequence of coefficients $C^{\pm}_{n,m} \in \mathbb{C}$. The sums converge in $H^1_{loc}(\Omega^+_b)$ (resp. $H^1_{loc}(\Omega^+_b)$).

Recall our definition of $H^1_{loc}(\Omega^{\pm}_b)$ as the space of all functions v over $\Omega^{\pm}_b$ such that, for any radius $r > 0$, the restriction of v to $\Omega^{\pm}_{b,r} := \{x \in \Omega^{\pm}_b : |x| < r\}$ is in $H^1(\Omega^{\pm}_{b,r})$. Note that the functions $u \in H^1_{loc}(\Omega^{\pm}_b)$ of Definition 4.18 satisfy the Helmholtz equation $\Delta u(x_1, x_2) + k^2 q(x_1) u(x_1, x_2) = 0$ for $(x_1, x_2) \in \Omega^{\pm}_b$.

If $q(x) \equiv q_0 \in \mathbb{C}$, the upward and downward propagating modes defined in Definitions 4.18 and 4.16 are exactly the Rayleigh modes occurring in a homogeneous periodic medium. In fact, the spectrum (μ_n, h_n) of the differential operator L is given by

$$\mu_n = \alpha_n^2 - k^2 q_0 \in \mathbb{C}, \qquad h_n(x_1) = \exp(i\alpha_n x_1), \quad n \in \mathbb{Z}.$$

In particular, each eigenvalue μ_n is of rank one and there is no associated eigenfunctions of rank $m \geq 2$ (see the arguments below). Correspondingly, the spectrum (λ_n, W_n) of the matrix differential operator M can be represented as (see Lemma 4.3)

$$\lambda_n^{\pm} = \pm\sqrt{\alpha_n^2 - k^2 q_0}, \quad W_n^{\pm} = \exp(i\alpha_n x_1) \begin{pmatrix} 1 \\ \pm\sqrt{\alpha_n^2 - k^2 q_0} \end{pmatrix}.$$

Note that the branch of $\sqrt{a}$ is taken such that $\operatorname{Im}\sqrt{a} \geq 0$ for $a \in \mathbb{C}$. By the definition (4.33), the parameter $\hat{\lambda}_n \in \mathbb{C}$ turns out to be

$$\hat{\lambda}_n := \begin{cases} -\sqrt{\alpha_n^2 - k^2 q_0} & \text{if} \quad |\alpha_n|^2 > |k^2 q_0|, \\ \sqrt{k^2 q_0 - \alpha_n^2} & \text{if} \quad |\alpha_n|^2 \leq |k^2 q_0|. \end{cases}$$

Hence, the upward and downward going modes take the form

$$u_n^{(U)}(x) = e^{i\alpha_n x_1 + \hat{\lambda}_n (x_2 - b)}, \quad x_2 \geq b,$$
$$u_n^{(D)}(x) = e^{i\alpha_n x_1 - \hat{\lambda}_n (x_2 - b)}, \quad x_2 \leq b.$$

In the special case $q(x) \equiv 1$, it holds that

$$\hat{\lambda}_n := \begin{cases} -\sqrt{\alpha_n^2 - k^2} & \text{if} \quad |\alpha_n| > k, \\ i\sqrt{k^2 - \alpha_n^2} & \text{if} \quad |\alpha_n| \leq k, \end{cases}$$

which coincides with $i\beta_n$ for any $n \in \mathbb{Z}$ (cf. (2.4)). If $\mu_n = 0$ is an eigenvalue of L, we have either $\alpha_n = k$ or $\alpha_n = -k$, that is, the dimension of the eigenspace $\sigma_L(0)$ is at most two, with the eigenfunctions $e^{\pm ikx_1}$. These eigenmodes can be regarded as both upward and downward going modes. When $\alpha_n = 0$ for some $n \in \mathbb{Z}$, it holds that $u_n^{(U)}(x) = e^{ikx_2}$ and $u_n^{(U)}(x) = e^{-ikx_2}$, which are 2π-periodic wave modes in the x_2-direction.

Next we show that the rank of the eigenvalue μ_n of the operator $L = -(\partial_1^2 + k^2 q_0 I)$ with $q_0 \in \mathbb{C}$ is at most one. For this purpose, it suffices to prove that, for any given $n \in \mathbb{N}$, there do not exist α-quasiperiodic solutions to the ordinary differential equation

$$w''(x_1) + \alpha_n^2 w(x_1) = e^{i\alpha_n x_1}, \quad x_1 \in \mathbb{R}. \tag{4.34}$$

If $\alpha_n \neq 0$, a general solution to (4.34) takes the form

$$\begin{aligned} w(x_1) &= c^+ e^{i\alpha_n x_1} + c^- e^{-i\alpha_n x_1} + v(x_1), \quad c^\pm \in \mathbb{C}, \\ v(x_1) &= \frac{1}{\alpha_n} \int_0^{x_1} \sin\big(\alpha_n(x_1 - y_1)\big) e^{i\alpha_n y_1}\, dy_1 \\ &= \frac{-e^{i\alpha_n x_1}}{4\alpha_n^2} \left(e^{-i2\alpha_n x_1} - 1 + i2\alpha_n x_1 \right). \end{aligned} \tag{4.35}$$

It is easy to see

$$v'(x_1) = \int_0^{x_1} \cos\left(\alpha_n(x_1 - y_1)\right) e^{i\alpha_n y_1}\, dy_1 = \frac{i\, e^{i\alpha_n x_1}}{4\alpha_n} \left(e^{-i2\alpha_n x_1} - 1 - i2\alpha_n x_1 \right) \tag{4.36}$$

and $v(0) = v'(0) = 0$. The function w is α-quasiperiodic in x_1 if $w(0) = w(2\pi)e^{-i2\pi\alpha}$ and $w'(0) = w'(2\pi)e^{-i2\pi\alpha}$. Since $e^{i\alpha_n x_1}$ is α-quasiperiodic, we get conditions on c^- and can assume $c^+ = 0$. The first condition together with $\alpha_n = \alpha + n$ leads us to $(c^- + v(0))e^{i\alpha 2\pi} = (c^- e^{-i\alpha 2\pi} + v(2\pi))$ i.e., to the formula $2i \sin(\alpha 2\pi) c^- = v(2\pi)$. Similarly, the second condition for the derivatives implies $2i \sin(\alpha 2\pi) c^- = \frac{i}{\alpha_n} v'(2\pi)$. In other words, an existence of a quasiperiodic solution (4.34) requires $v(2\pi) = \frac{i}{\alpha_n} v'(2\pi)$. Substituting $x_1 = 2\pi$ into the formulas (4.35) and (4.36), we get $\alpha_n = 0$, which is a contradiction to the assumption $\alpha_n \neq 0$ for our case. If $\alpha_n = 0$, it holds that $\alpha = -n$ for some $n \in \mathbb{Z}$, implying that the solution w to the ordinary equation $w'' = 1$ must be 2π-periodic. A general solution of (4.34) is given by $w(x_1) = 1/2\, x_1^2 + a x_1 + b$ with $a, b \in \mathbb{C}$. However, such general solutions cannot be 2π-periodic. In summary, eigenvalues for constant potentials cannot be of rank $m \geq 2$.

5. Solvability of grating diffraction problems in an inhomogeneous periodic medium

The derivation of the results on the solvability of the boundary value problem, modeling the scattering of an incoming wave by the grating structure between inhomogeneous media, goes along the same lines as in the case of homogeneous cover and substrate materials. In Subsect. 5.1, we shall define Dirichlet-to-Neumann (DtN) mappings over the lower boundary line of the cover material and over the upper boundary of the substrate. Mapping properties of these DtN operators will be investigated in Lemmata 5.3, 5.4 and 5.5. In particular, definiteness and strong ellipticity of the quadratic forms corresponding to the two Dirichlet-to-Neumann mappings are presented. In Sect. 5.2, we formulate the scattering problem as a quasiperiodic boundary value problem. An equivalent variational formulation is given by enforcing the Dirichlet-to-Neumann mappings on an artificial boundary inside the inhomogeneous material, and the strong ellipticity of the corresponding sesquilinear form is proved. The definiteness of the quadratic forms imply the uniqueness of the scattered far-field, namely the reflected and transmitted propagating wave modes. By Fredholm's alternative, we obtain unique solvability of the scattering problem for absorbing materials and also existence of solutions in non-absorbing materials for special incoming waves.

In the subsequent sections we suppose that $q(x) = q(x_1) \in L^\infty(\mathbb{R})$ is complex-valued, 2π-periodic with respect to x_1 and satisfies Assumption RC(q) of Definition 4.15. We extend q from Ω_b^- to $\mathbb{R}^2$ by setting $q(x) = q(x_1)$ for all $x \in \mathbb{R}^2$.

5.1. Dirichlet-to-Neumann mappings

Similarly to Subsect. 4.4 and in contrast to the notation h_n, $n \in \mathbb{N}_0$ for the system of eigenfunctions and associated eigenfunctions used in Subsect. 4.2 (cf. Lemma 4.6), we again denote this system by $h_{n,m}$, $(n,m) \in I$ with $I := \{(n,m) : n \in \mathbb{N},\, m \in A_L^F(\mu_n)\}$. The index m denotes the rank of the associate eigenfunction $h_{n,m} \in A_L(\mu_n)$ for the eigenvalue μ_n introduced after Lemma 4.8. In the subsequent sections we identify the straight line Γ_b with the finite section over a single period $\{(x_1, b) : x_1 \in (0, 2\pi)\}$. For $d > b$, we define the rectangular domain $R_{b,d} := \{x \in \mathbb{R}^2 : b < x_2 < d,\ 0 < x_1 < 2\pi\}$. Hence, $\Gamma_b \cup \Gamma_d$ is a subset of the boundary of $R_{b,d}$.

Lemma 5.1. *The system $h_{n,m}$, $(n,m) \in I$ is complete in $H_\alpha^{1/2}(\Gamma_b)$. If it is a Riesz basis in $L^2(\Gamma_b)$, then a scaled version of the system is a Riesz basis in $H_\alpha^{1/2}(\Gamma_b)$.*

Proof. In accordance with Lemma 4.6 the linear span of the system $h_{n,m}$, $(n,m) \in I$ is dense in $L^2(\Gamma_b)$. Using that $L^2(\Gamma_b)$ is a dense subspace in $H_\alpha^{-1}(\Gamma_b)$, we conclude that the span of system $h_{n,m}$, $(n,m) \in I$ is dense in $H_\alpha^{-1}(\Gamma_b)$ as well. Now, knowing that $q \in L^\infty$, we can choose a real number κ such that $A := L + \kappa I : H_\alpha^1(\Gamma_b) \to H_\alpha^{-1}(\Gamma_b)$ is invertible. Then the span of system $A^{-1}h_{n,m}$, $(n,m) \in I$ is dense in $H_\alpha^1(\Gamma_b)$. However, the $h_{n,m}$ are eigenfunctions or associate eigenfunctions of operator A. Consequently, the span of system $A^{-1}h_{n,m}$, $(n,m) \in I$ coincides with the span of the system $h_{n,m}$, $(n,m) \in I$. In other words, the span of system $h_{n,m}$, $(n,m) \in I$ is dense in $H_\alpha^1(\Gamma_b)$. Since $H_\alpha^1(\Gamma_b)$ is dense in $H_\alpha^{1/2}(\Gamma_b)$, the span of system $h_{n,m}$, $(n,m) \in I$ is dense in $H_\alpha^{1/2}(\Gamma_b)$. The Riesz basis property follows from Lemma 4.8. □

Definition 5.2. The Dirichlet-to-Neumann maps $\mathcal{T}_b^\pm$ for upward and downward radiating solutions are defined as

$$\mathcal{T}_b^\pm(f) := \pm(\partial_2 u_\pm^{sc})|_{\Gamma_b}, \qquad f \in H_\alpha^{1/2}(\Gamma_b),$$

where $u_\pm^{sc}$ are the upward and downward radiating solutions to the Dirichlet boundary value problem

$$\Delta u_\pm^{sc} + k^2 q u_\pm^{sc} = 0 \quad \text{for} \qquad x_2 \geq b\ (x_2 \leq b), \qquad u_\pm^{sc}|_{\Gamma_b} = f. \tag{5.1}$$

Given $f \in H_\alpha^{1/2}(\Gamma_b) \subset L_\alpha^2(\Gamma_b)$, by Lemmas 4.6 and 4.8 we may expand f into the series

$$f = \sum_{(n,m)\in I} f_{n,m}\, h_{n,m}, \quad f_{n,m} := \langle f, h_{n,m}^* \rangle \in \mathbb{C}, \tag{5.2}$$

where $\{h_{n,m}^*\}$ is the dual system of $\{h_{n,m}\}$. Recall the equivalent norm (cf. Lemma 4.8 valid for the Riesz basis $h_{n,m}$, $(n,m) \in I$)

$$||f||^2_{H_\alpha^{1/2}(\Gamma_b)} \sim \sum_{(n,m)\in I} (1 + |n|)\, |f_{n,m}|^2 + \sum_{(n,m)\in I_d} (1 + \kappa_{n,m})^{1/2}\, |f_{n,m}|^2.$$

Using Theorem 4.10, the solution $u_\pm^{sc} \in H_{loc}^1(\Omega_b^\pm)$ to the boundary value problem (5.1) takes the form

$$u_+^{sc} = \sum_{(n,m)\in I} f_{n,m}\, u_{n,m}^{(U)}, \quad x_2 \geq b, \tag{5.3}$$

$$u_-^{sc} = \sum_{(n,m)\in I} f_{n,m}\, u_{n,m}^{(D)}, \quad x_2 \leq b. \tag{5.4}$$

Lemma 5.3. *Suppose Assumption RC(q) given in Definition 4.15. Then the sums in (5.3) and (5.4) converge in* $H^1_{loc}(\Omega_b^+)$*, and the mappings* $\mathcal{T}_b^\pm$ *are continuous from* $H_\alpha^{1/2}(\Gamma_b)$ *to* $H_\alpha^{-1/2}(\Gamma_b)$*.*

Proof. Without loss of generality we consider the case of $+$ and upgoing waves. Any approximation of $\mathcal{T}_b^+$, defined by a finite section of the index set, is obviously continuous. Thus, due to Lemma 4.5 (iv), we may suppose that all $h_{n,m}$ are eigenfunctions of rank one or two for eigenvalues μ_n with $\operatorname{Re}\mu_n > 0$. First we assume that all these eigenfunctions are of rank one. We fix a small $\varepsilon_D > 0$. If $h_{n,1}^*$ is a function in the dual system, then

$$T_{co} f(x_1) := u_+^{sc}(x_1, b+\varepsilon_D) = \sum_n \langle f, h_{n,1}^* \rangle u_{n,1}^{(U)}(x_1, b+\varepsilon_D).$$

We assume that the sum contains only a finite number of terms. From Lemma 4.5, (iv) and $u_{n,1}^{(U)}(x_1,x_2) = \exp\big(-\sqrt{\mu_n}(x_2-b)\big)h_{n,1}(x_1)$, we obtain

$$|u_+^{sc}(x_1, b+\varepsilon_D)| \leq c \sum_n \|f\|_{L^2(\Gamma_b)} \exp[-\operatorname{Re}\sqrt{\mu_n}\,\varepsilon_D\,] \;\leq\; c\|f\|_{H_\alpha^{1/2}(\Gamma_b)}.$$

Similarly, we can estimate $|\partial_{x_1}^2 u_+^{sc}(x_1, b+\varepsilon_D)|$ if we use that $h_{n,1}$ is an eigenfunction of L. We arrive at

$$\|T_{co} f\|_{H_\alpha^{1/2}(\Gamma_b)} = \|u_+^{sc}|_{\Gamma_{b+\varepsilon_D}}\|_{H_\alpha^{1/2}(\Gamma_b)} \leq c\|f\|_{H_\alpha^{1/2}(\Gamma_b)}.$$

Now we use the continuity of the Dirichlet problem for α-quasiperiodic Helmholtz solutions in the rectangle $R_{b,b+\varepsilon_D}$. For sufficiently small ε_D, the variational form $(u,v) \mapsto -\int \nabla u \cdot \nabla \hat{v} + k^2 \int q u \hat{v}$ of the quasiperiodic Dirichlet problem

$$\Delta u(x) + k^2 q(x_1) u(x) = 0,\ x \in R_{b,b+\varepsilon_D}, \quad u|_{\Gamma_b} = f,\ u|_{\Gamma_{b+\varepsilon_D}} = f_2 \tag{5.5}$$

is coercive over the space of functions $u \in H_\alpha^1(\mathbb{R}_{b,b+\varepsilon_D})$ with $u|_{\Gamma_b} = 0$ and $u|_{\Gamma_{b+\varepsilon_D}} = 0$. We denote the solution of (5.5) by $U[f, f_2]$ and get

$$\|U[f,f_2]\|_{H_\alpha^1(R_{b,b+\varepsilon_D})} \leq c\|f\|_{H_\alpha^{1/2}(\Gamma_b)} + c\|f_2\|_{H_\alpha^{1/2}(\Gamma_{b+\varepsilon_D})},$$

as well as $U[f, f_2] = u_+^{sc}|_{R_{b,b+\varepsilon_D}}$. We conclude

$$\begin{aligned}\|\mathcal{T}_b^+ f\|_{H^{-1/2}(\Gamma_b)} &\leq c\|U[f, T_{co}f]\|_{H^1(R_{b,b+\varepsilon_D})} \;\leq\; c\left\{\|f\|_{H_\alpha^{1/2}(\Gamma_b)} + \|T_{co}f\|_{H_\alpha^{1/2}(\Gamma_{b+\varepsilon_D})}\right\}\\ &\leq c\|f\|_{H_\alpha^{1/2}(\Gamma_b)}.\end{aligned}$$

Consequently, we can extend $\mathcal{T}_b^+$ to a continuous operator over $H_\alpha^{1/2}(\Gamma_b)$, and the sum (5.3) converges in $H_\alpha^1(R_{b,b+\varepsilon_D})$. Similarly, we get convergence and boundedness in $H_\alpha^1(R_{b+\varepsilon_D,b+2\varepsilon_D})$, in $H_\alpha^1(R_{b+2\varepsilon_D,b+3\varepsilon_D})$, and so on. In other words, we get convergence in $H_{loc}^1(\Omega_b^+)$.

If there exist rank-two eigenfunctions in the sum, then we can proceed similarly. We only have to use Corollary 4.11 together with (4.16) and $c_{n,k,j} = O(|n|^4)$, $k, j = 1, 2$, which has been shown at the end of the proof to Lemma 4.8. □

Below we investigate other properties of $\mathcal{T}_b^\pm$. In contrast to the orthogonal basis $e^{i\alpha_n x_1}$ (identical with its dual system) for a homogeneous medium, the Riesz bases $h_{n,m}$ in our case may not be orthogonal. The following two lemmas for the homogeneous case were justified in a straightforward manner by the definition of DtN mappings. As we shall show, their generalization to media with non-constant but real-valued q is easy. In this paper we shall make use of variational arguments to prove them even for complex-valued q.

Lemma 5.4. *Suppose Assumption RC(q) given in Definition 4.15 and let $f \in H_\alpha^{1/2}(\Gamma_b)$ be given by (5.2) with coefficients $f_{n,m} \in \mathbb{C}$.*

(i) *For real-valued q, each mode $u_{n,m}^{(U)}$ (resp. $u_{n,m}^{(D)}$) corresponds to associate eigenfunctions of rank one, i.e., $m = 1$. Furthermore, we have*

$$\operatorname{Im} \int_{\Gamma_b} \mathcal{T}_b^\pm f \, \bar{f} \geq 0 \quad \textit{for all} \quad f \in H_\alpha^{1/2}(\Gamma_b). \tag{5.6}$$

If the equality sign in (5.6) holds, then we have $f_{n,1} = 0$ for all n with $\operatorname{Im} \hat{\lambda}_n > 0$, that is, the solution to the boundary value problem (2.5) has no propagating wave mode with $\operatorname{Re} \hat{\lambda}_n = 0$ and $\operatorname{Im} \hat{\lambda}_n > 0$.

(ii) *If $\operatorname{Im} q \geq c_q > 0$ on a subdomain, then there is no propagating mode. Moreover, the inequality (5.6) still holds, and, in the case of equality sign, we have $f_{n,m} = 0$ for all $(n,m) \in I$.*

Proof. We consider $\mathcal{T}_b^+$ and the upward radiating modes only. The case of $\mathcal{T}_b^-$ can be treated analogously.

(i) For real-valued q, we have a self-adjoint operator, and there is no $h_{n,m}$ with rank m greater than one. Moreover, the eigenfunctions are orthogonal. Choosing a sufficiently large n_0 and substituting

$$(\mathcal{T}_b^+ f)(x_1) = \sum_{n \in \mathbb{Z}} \hat{\lambda}_n \, f_{n,1} \, h_{n,1}(x_1)$$

into (5.6), the assertion follows from $\operatorname{Im} \hat{\lambda}_n \geq 0$ and the identity

$$\operatorname{Im} \int_{\Gamma_b} \mathcal{T}_b^+ f \, \bar{f} \, ds = \sum_{n \in \mathbb{Z}} (\operatorname{Im} \hat{\lambda}_n) \, |f_{n,1}|^2 \int_0^{2\pi} |h_{n,1}(x_1)|^2 \, dx_1 \geq 0.$$

(ii) Now consider the boundary value problem (2.5) in $x_2 \geq b$ and suppose $\operatorname{Im} q > 0$ on a set of positive measure. Equ. (5.3) together with Green's formula leads us to

$$\int_{\Gamma_b} \mathcal{T}_b^+ f\,\bar{f}\,ds = \int_{\Gamma_d} \partial_{x_2} u_+^{sc} \bar{u}_+^{sc}\,ds + \int_{R_{b,d}} \left\{ k^2 q\,|u_+^{sc}|^2 - |\nabla u_+^{sc}|^2 \right\} dx. \tag{5.7}$$

To prove that there is no propagating mode, we only need to consider a propagating mode of rank one. Taking $f := \tilde{h}_n$ with $\operatorname{Re}\hat{\lambda}_n = 0$, we get $\operatorname{Im}\hat{\lambda}_n \geq 0$ and

$$\begin{aligned} u_+^{sc}(x) &= e^{\hat{\lambda}_n(x_2-b)}\,\tilde{h}_n(x_1), && \text{in} \quad x_2 \geq b, \\ \partial_2 u_+^{sc}(x) &= \hat{\lambda}_n\, e^{\hat{\lambda}_n(d-b)}\,\tilde{h}_n(x_1), && \text{on} \quad x_2 = d. \end{aligned}$$

Taking the imaginary part of (5.7) and using $q = q(x_1)$ we get

$$\begin{aligned} \operatorname{Im} \int_{\Gamma_b} \mathcal{T}_b^+ \tilde{h}_n\,\overline{\tilde{h}_n} ds &= k^2 \int_{R_{b,d}} \operatorname{Im}(q) |u_+^{sc}|^2\,dx + \operatorname{Im}(\hat{\lambda}_n) \int_{\Gamma_d} |\tilde{h}_n|^2\,ds \\ &= k^2(d-b) \int_0^{2\pi} \operatorname{Im}(q) |\tilde{h}_n|^2\,dx_1 + \operatorname{Im}(\hat{\lambda}_n) \int_0^{2\pi} |\tilde{h}_n|^2\,dx_1, \end{aligned}$$

for any $d > b$. We conclude that $\int_0^{2\pi} \operatorname{Im}(q)|\tilde{h}_n|^2\,dx_1 = 0$, since the right-hand side should be independent of $d > b$. Hence, $\tilde{h}_n(x_1) = 0$ over the subdomain where $\operatorname{Im} q(x_1) \geq c_q$. This further yields $u_+^{sc} \equiv 0$ in $x_2 \geq b$ by unique continuation of the elliptic equation (see e.g., [20, Thm. 17.2.6, Chapter XVII]) and thus $\tilde{h}_n \equiv 0$.

Next we shall prove the inequality (5.6) for complex-valued $q(x_1)$. For $f = \sum_{n,m} f_{n,m} h_{n,m}$, the solution u_+^{sc} is given by (5.3). As $d \to \infty$, the exponentially decaying terms $u_{n,m}^{(U)}(x_1, d)$ with $\operatorname{Re}\hat{\lambda}_n = 0$ tend to zero, and only the propagating modes remain. Hence

$$u_+^{sc}(x_1, d) \to \sum_{(n,m)\in I : \operatorname{Re}\hat{\lambda}_n = 0} \hat{\lambda}_n\, f_{n,m}\, u_{n,m}^{(U)}(x_1, d) = 0, \quad \text{as} \quad d \to \infty.$$

In the last step, we have used the vanishing of the propagating modes, that is, $u_{n,m}^{(U)} \equiv 0$ if $\operatorname{Re}\hat{\lambda}_n = 0$. Similarly, one can prove that $\partial_2 u_+^{sc}(x_1, d) \to 0$ as $d \to \infty$. Taking the imaginary part of (5.7) and letting $d \to \infty$, we obtain

$$\operatorname{Im} \int_{\Gamma_b} \mathcal{T}_b^+ f\,\bar{f}\,ds = \lim_{d\to\infty} \left\{ \int_{R_{b,d}} k^2 [\operatorname{Im} q]\,|u_+^{sc}|^2\,dx \right\} \geq 0.$$

In the case of equality sign, we must have $u_+^{sc} \equiv 0$ and thus $f_{n,m} = 0$ for all $(n,m) \in I$. □

Lemma 5.5. *Suppose there holds Assumption RC(q) given in Definition 4.15. Then there exists a compact operator* $\mathcal{T}_{b,0}^{\pm} : H_\alpha^{1/2}(\Gamma_b) \to H_\alpha^{-1/2}(\Gamma_b)$ *such that*

$$\int_0^{2\pi} \left[-\mathcal{T}_b^{\pm} + \mathcal{T}_{b,0}^{\pm}\right] f\,\bar{f}\,ds \geq c_0\,||f||^2_{H_\alpha^{1/2}(\Gamma_b)}, \quad c_0 > 0.$$

In other words, $-\mathcal{T}_b^{\pm}$ *can be decomposed into the sum of a coercive operator and a compact operator.*

Proof. The assertions for $\mathcal{T}_b^+$ and $\mathcal{T}_b^-$ follow analogously. So we only consider the case of $\mathcal{T}_b^+$. For $d > b$, the identity (5.7) can be decomposed into two parts:

$$-\int_{\Gamma_b} \mathcal{T}_b^+ f\,\bar{f}\,ds = \int_{R_{b,d}} \left\{|\nabla u_+^{sc}|^2 + |u_+^{sc}|^2\right\} dx - \int_{\Gamma_b} \mathcal{T}_{b,0}^+ f\,\bar{f}\,ds \tag{5.8}$$

where $\mathcal{T}_{b,0}^+ : H_\alpha^{1/2}(\Gamma_b) \to H_\alpha^{-1/2}(\Gamma_b)$ is defined as

$$\int_{\Gamma_b} \mathcal{T}_{b,0}^+ f\,\bar{g}\,ds := \int_{R_{b,d}} \left\{(1+k^2 q)\,u_+^{sc}\,\bar{w}_+^{sc}\right\} dx + \int_{\Gamma_d} \partial_2 u_+^{sc}\,\bar{w}_+^{sc}\,ds, \quad g \in H_\alpha^{1/2}(\Gamma_b).$$

Here $w_+^{sc} \in H^1(R_{b,d})$ is the unique radiating solution to the boundary value problem (5.1) with the Dirichlet data $w_+^{sc} = g$ on Γ_b. The operator $\mathcal{T}_{b,0}^+$ is compact, because the mappings

$$\begin{aligned} G_1 &: H_\alpha^{1/2}(\Gamma_b) \to H_\alpha^{1/2}(\Gamma_d), \qquad G_1(g) := w_+^{sc}|_{\Gamma_d}, \\ G_2 &: H_\alpha^{1/2}(\Gamma_b) \to L_\alpha^2(R_{b,d}), \qquad G_2(g) := w_+^{sc}|_{R_{b,d}}, \end{aligned}$$

are both compact. On the other hand, by (5.8) it is clear that $-\mathcal{T}_b^+ + \mathcal{T}_{b,0}^+$ is a coercive operator on $H_\alpha^{1/2}(\Gamma_b)$. □

5.2. *Well-posedness of the transmission problem*

Next we consider the boundary value problem for the simulation of waves scattered at a grating located between the two inhomogeneous half spaces Ω_d^+ and Ω_b^- with $b < d$ (cf. Fig. 2). In particular, we assume $\tilde{q} \in L^\infty(\mathbb{R}^2)$ such that $\tilde{q}(x) = q^+(x_1)$ for $x_2 \geq d$ and $\tilde{q}(x) = q^-(x_1)$ for $x_2 \leq b$. In other words, the univariate function previously denoted by q is now changed to $q^\pm$. Of course, for the refractive index, we suppose there is a constant $c_q > 0$ such that either $\tilde{q}(x) > c_q$ or $\operatorname{Im}\tilde{q}(x) > c_q$. By $L_{b,d}$ we denote the layer $\{x \in \mathbb{R}^2 : b < x_2 < d\}$ and, as before, by $R_{b,d}$ the rectangle $\{x \in \mathbb{R}^2 : b < x_2 < d, 0 < x_1 < 2\pi\}$. For any given functions $f_D^d \in H_\alpha^{1/2}(\Gamma_d)$, $f_N^d \in H_\alpha^{-1/2}(\Gamma_d)$, $f_D^b \in H_\alpha^{1/2}(\Gamma_b)$, and $f_N^b \in H_\alpha^{-1/2}(\Gamma_b)$, we look for a triple of α-quasiperiodic field solutions $u \in H_\alpha^1(L_{b,d})$, $u^+ \in H^1_{\alpha,loc}(\Omega_d^+)$, and $u \in H^1_{\alpha,loc}(\Omega_b^-)$ of

$$\begin{aligned} \Delta u(x) \quad + k^2\tilde{q}(x) \quad u(x) \quad &= 0, \quad x \in L_{b,d}, \\ \Delta u^+(x) + k^2 q^+(x_1) u^+(x) &= 0, \quad x \in \Omega_d^+, \\ \Delta u^-(x) + k^2 q^-(x_1) u^-(x) &= 0, \quad x \in \Omega_b^-, \end{aligned}$$

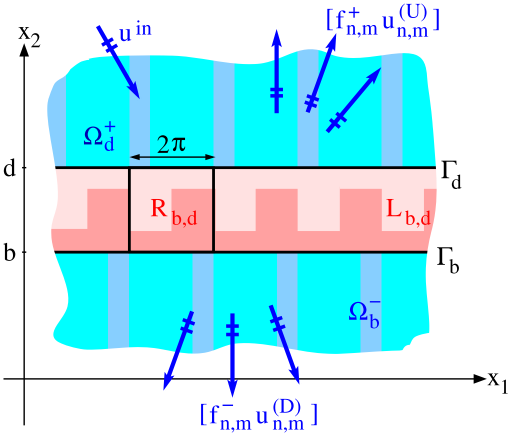

Fig. 2. The geometry settings for the boundary value problem.

$$\begin{aligned} u|_{\Gamma_d} &= u^+|_{\Gamma_d} + f_D^d, \quad \partial_2 u|_{\Gamma_d} = \partial_2 u^+|_{\Gamma_d} + f_N^d, \\ u|_{\Gamma_b} &= u^-|_{\Gamma_b} + f_D^b, \quad \partial_2 u|_{\Gamma_b} = \partial_2 u^-|_{\Gamma_b} + f_N^b, \\ &u^+ \text{ is an upward radiating wave in } \Omega_d^+, \\ &u^- \text{ is a downward radiating wave in } \Omega_b^-. \end{aligned} \tag{5.9}$$

Suppose that $u^{in} \in H^1_{\alpha,loc}(\Omega_d^+)$ is a downward incoming wave satisfying the Helmholtz equation $(\Delta + k^2 q^+ I)u^{in} = 0$ in Ω_d^+. Then the wave solution of (5.9) with $f_D^d = u^{in}|_{\Gamma_d}$, $f_N^d = \partial_{x_2} u^{in}|_{\Gamma_d}$, $f_D^b = 0$, and $f_N^b = 0$ is the wave scattered by the grating, i.e., u^+ is the reflected wave, u^- the transmitted wave, and u the wave induced inside the grating.

Clearly, the weak formulation of (5.9) is the variational equation

$$\begin{aligned} a(u,v) &= F(v), \ \forall v \in H^1_\alpha(R_{b,d}), \\ a(u,v) &:= \int_{R_{b,d}} \left\{ -\nabla u \cdot \nabla \bar{v} + k^2 \tilde{q}\, u\, \bar{v} \right\} dx + \int_{\Gamma_d} \mathcal{T}_d^+ u\, \bar{v}\, ds + \int_{\Gamma_b} \mathcal{T}_b^- u\, \bar{v}\, ds, \\ F(v) &:= \int_{\Gamma_d} \left[\mathcal{T}_d^+ f_D^d - f_N^d \right] \bar{v}\, ds + \int_{\Gamma_b} \left[\mathcal{T}_b^- f_D^b + f_N^b \right] \bar{v}\, ds. \end{aligned} \tag{5.10}$$

The variational solution $u \in H^1_\alpha(R_{b,d})$ can be extended to Ω_d^+ and Ω_b^- as follows. If u is the weak solution, then we get $u|_{\Gamma_d} - f_D^d = \sum_{n,m} f^+_{n,m} h_{n.m}$ with coefficients $f^+_{n,m} \in \mathbb{C}$ and the eigenfunction $h_{n,m} = h_{n,m}(\Omega_d^+)$ for the domain Ω_d^+. We get the solution for $x_2 > d$ by the extension $u^+ = \sum_{n,m} f^+_{n,m} u^{(U)}_{n.m}$. For $x_2 < b$, we get $u|_{\Gamma_b} - f_D^b = \sum_{n,m} f^-_{n,m} h_{n.m}$ with $f^-_{n,m} \in \mathbb{C}$ and the eigenfunction $h_{n,m} = h_{n,m}(\Omega_b^-)$ for Ω_b^-. The solution for $x_2 < b$ is the extension $u^- = \sum_{n,m} f^-_{n,m} u^{(D)}_{n.m}$.

Now we prepare the solvability theorem by

Lemma 5.6. *Suppose the Assumptions RC($q^{\pm}$) introduced in Definition 4.15 hold. The sesquilinear form $a : H^1_\alpha(R_{b,d}) \times H^1_\alpha(R_{b,d}) \to \mathbb{R}$ is bounded. Moreover, it is strongly elliptic, i.e., there exists a compact operator $T_{se} : H^1_\alpha(R_{b,d}) \to H^{-1}_\alpha(R_{b,d})$ and a constant $c_{se} > 0$ such that, for all $u \in H^1_\alpha(R_{b,d})$,*

$$|a(u,u) + \langle T_{se}u, u\rangle| \geq c_{se}\|u\|^2_{H^1_\alpha(R_{b,d})},$$

where $\langle v, u\rangle$ denotes the duality pairing between $H^{-1}_\alpha(R_{b,d})$ and $H^1_\alpha(R_{b,d})$, which is equal to the L^2 scalar product for $v \in L^2(R_{b,d})$. The right-hand side functional $F : H^1_\alpha(R_{b,d}) \to 0$ is continuous.

Proof. The boundedness follows from Lemma 5.3, the strong ellipticity from Lemma 5.4. The continuity of F is a consequence of Lemma 5.3. □

Theorem 5.7. *Suppose the Assumptions RC($q^{\pm}$) introduced in Definition 4.15 hold.*

(i) *The space of all weak solutions to the homogeneous boundary value problem (5.9) with $f^d_D = f^b_D = f^d_N = f^b_N = 0$ has a finite dimension. The space of homogeneous solutions of the adjoint differential operator, i.e.,*

$$\ker := \left\{ v \in H^1_\alpha(R_{b,d}) : a(w,v) = 0,\ \forall w \in H^1_\alpha(R_{b,d}) \right\}$$

has the same finite dimension. There exists a weak solution of (5.9) if and only if, for any $v \in \ker$, the condition $F(v) = 0$ holds. If this solvability condition is satisfied and if u_p is a particular solution of (5.9), then the general weak solution is $u = u_p + u_h$ with u_h a weak solution of the homogeneous boundary value problem (5.9).

(ii) *Assume the function q^+ is real-valued and let $\hat{\lambda}_{n_0} = \hat{\lambda}_{n_0}(\Omega^+_d)$ be defined as in (4.33) such that $\operatorname{Re}\hat{\lambda}_{n_0} = 0$, $\operatorname{Im}\hat{\lambda}_{n_0} > 0$. Suppose that the incoming wave u^{in} in Ω^+_d is the propagating downward radiating mode $u^{in} = u^{(D)}_{n_0,1}(\Omega^+_d)$. Then there exists a weak solution of (5.9) with $f^d_D = u^{in}|_{\Gamma_d}$, $f^d_N = \partial_2 u^{in}|_{\Gamma_d}$ and $f^b_D = f^b_N = 0$.*

(iii) *For real-valued squared refractive index $q^{\pm}$, the propagating upward (resp. downward) radiating modes in Ω^+_d (resp. Ω^-_b) with $\operatorname{Re}\hat{\lambda}_n = 0$ and $\operatorname{Im}\hat{\lambda}_n > 0$ for the general boundary value problem (5.9) are uniquely determined.*

(iv) *Suppose that $\operatorname{Im}\tilde{q}(x) \geq c_{\tilde{q}} > 0$ over a subdomain $D_0 \subset R_{b,d}$ or that $\operatorname{Im} q^{\pm}(x_1) \geq c_{q^{\pm}} > 0$ over a subinterval of $[0, 2\pi]$. Then there exists a unique weak solution u of (5.9), and for a constant $C_s > 0$ independent of the boundary data f^d_D, f^d_N, f^b_D and f^b_N, we get*

$$\begin{aligned}&\|u\|_{H^1_\alpha(R_{b,d})} + \|u^+|_{\Gamma_d}\|_{H^{1/2}_\alpha(\Gamma_d)} + \|u^-|_{\Gamma_b}\|_{H^{1/2}_\alpha(\Gamma_b)} \\ &\quad \leq C_s \left\{ \|f^d_D\|_{H^{1/2}_\alpha(\Gamma_d)} + \|f^d_N\|_{H^{-1/2}_\alpha(\Gamma_d)} + \|f^b_D\|_{H^{1/2}_\alpha(\Gamma_b)} + \|f^b_N\|_{H^{-1/2}_\alpha(\Gamma_b)} \right\}\end{aligned}$$

Proof. (i) Clearly, part (i) is a simple consequence of Fredholm's alternative applied to the variational equation (5.10), the sesquilinear form of which is strongly elliptic due to Lemma 5.6.

(ii) We apply (i). Suppose $v \in \ker$ is a solution of the homogeneous adjoint equation. Then we get $\operatorname{Im} a(v,v) = 0$. Using $\operatorname{Im}\int k^2\tilde{q}v\bar{v} \geq 0$ and Lemma 5.4 over Ω^-_b, we get $\operatorname{Im}\int_{\Gamma_d} \mathcal{T}^+_b v\,\bar{v} = 0$.

In the case of real-valued q^+, the eigenfunctions have rank one and form an orthogonal basis. There is a finite number of eigenvalues $\hat{\lambda}_n$ with $\operatorname{Re}\hat{\lambda}_n = 0$, and the remaining eigenvalues satisfy $\operatorname{Re}\hat{\lambda}_n > 0$. Thus, for $v = \sum_n f_{n,1} h_{n,1}$ it follows from Lemma 5.4 (i) that all propagating modes must vanish, i.e., $f_{n,1} = 0$ for $\operatorname{Im}\hat{\lambda}_n > 0$. In particular, we have $f_{n_0,1} = 0$. Hence, by the choice of the f_D^d, f_N^d, f_D^b, f_N^b and the orthogonality of $h_{n,m}$ we obtain

$$F(v) = \int_{\Gamma_d} \left[\mathcal{T}_d^+ h_{n_0,1} - h_{n_0,1}\right] \bar{v}\, ds = (\hat{\lambda}_{n_0} - 1)\, \bar{f}_{n_0,1} \int_0^{2\pi} |h_{n_0,1}|^2 dx_1 = 0.$$

The solution exists by Fredholm's alternative in part (i) of the lemma.

(iii) As shown in the proof of (ii), it follows from the variational formulation for the homogeneous boundary value problem that

$$\operatorname{Im} \int_{\Gamma_d} \mathcal{T}_d^+ u^+ \, \overline{u^+}\, ds + \operatorname{Im} \int_{\Gamma_b} \mathcal{T}_b^- u^- \, \overline{u^-}\, ds = 0,$$

which together with Lemma 5.4 (i) proves the assertion.

(iv) We have to show that any weak solution u of the homogeneous problem is identically zero. From the variational equation (5.10) we conclude $\operatorname{Im} a(u,u) = 0$ and thus

$$0 = \operatorname{Im} a(u,u) \geq \int_{D_0} k^2 \operatorname{Im} q\, |u|^2\, dx + \operatorname{Im} \int_{\Gamma_d} \mathcal{T}_d^+ u\, \bar{u}\, ds + \operatorname{Im} \int_{\Gamma_b} \mathcal{T}_b^- u\, \bar{u}\, ds \geq 0.$$

Applying Lemma 5.4 gives $u \equiv 0$ over D_0 if $\operatorname{Im} q(x) \geq c_{\tilde{q}} > 0$ in D_0. Hence, by unique continuation we get $u \equiv 0$ over $R_{b,d}$ (see [20, Thm. 17.2.6, Chapter XVII]). The case of $\operatorname{Im} q^{\pm}(x_1) \geq c_{q^{\pm}} > 0$ over a subinterval of $[0, 2\pi]$ can be proved analogously by applying Lemma 5.4 (ii). □

Remark 5.8. Equivalently, we could have formulated the theorem with the data f_D^d, f_N^d and f_D^b, f_N^b restricted to the subspace of traces $v^-|_{\Gamma_d}$, $\partial_2 v^-|_{\Gamma_d}$ of downward radiating waves v^- and to the subspace of traces $v^+|_{\Gamma_b}$, $-\partial_2 v^+|_{\Gamma_b}$ of upward radiating waves v^-, respectively. Indeed, the problem is linear such that the solution for general data is the superposition of solutions corresponding to the data given as traces of upward and downward radiating waves. However, the solution for $f_D^b = 0 = f_N^b$ and $f_D^d = v^+|_{\Gamma_d}$, $f_N^d = \partial_2 v^+|_{\Gamma_d}$ with v^+ an upward radiating wave is simply $u = 0 = u^-$ and $u^+ = v^+$. Similarly, the solution for $f_D^d = 0 = f_N^d$ and $f_D^b = v^-|_{\Gamma_d}$, $f_N^b = \partial_2 v^-|_{\Gamma_d}$ with v^- a downward radiating wave is simply $u = 0 = u^+$ and $u^- = v^-$.

Data availability

Data will be made available on request.

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