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# Increasing stability for inverse acoustic source problems in the time domain

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## Abstract

This paper is concerned with the inverse source problems for the acoustic wave equation in the full space $\mathbb{R}^3$, where the source term is compactly supported in both time and spatial variables. The main goal is to investigate increasing stability for the wave equation in terms of the interval length of given parameters (e.g., frequency bandwith of the temporal component of the source function). We establish increasing stability estimates of the $L^2$-norm of the source function by using only the Dirichlet boundary data. Our method relies on the Huygens' principle, the Fourier transform and explicit bounds for the continuation of analytic functions.


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## 1. Introduction

Consider an acoustic wave propagation problem caused by a compactly supported source function in three dimensions. This can be modeled by the time-dependent wave equation

$$\partial_t^2 u(x,t) - \lambda \Delta u(x,t) = F(x,t), \qquad x \in \mathbb{R}^3,\ t > 0, \tag{1.1}$$

where $u$ denotes the wave field, $0 < \lambda \in \mathbb{R}$ and $F$ is the source term. Together with the above governing equation, we impose the homogeneous initial conditions

$$u(x,0) = \partial_t u(x,0) = 0, \qquad x \in \mathbb{R}^3. \tag{1.2}$$

If $F \in L^2([0,\infty); L^2(\mathbb{R}^3))$ has a compact support, the problem (1.1)-(1.2) admits a unique solution (see e.g., [18,24,25])

$$u(x,t) \in \mathcal{C}([0,+\infty); H^1(\mathbb{R}^3)) \cap \mathcal{C}^1([0,+\infty); L^2(\mathbb{R}^3)).$$

In this paper we consider three inverse problems depending on the a priori knowledge on the parameter $\lambda$ and the form of the source function $F$.

**(IP1):** Assume $\lambda = 1$ and $F(x,t) = f(x)g(t)$, where the temporal function $g$ is given. Suppose that $f$, $g$ have compact support such that $\mathrm{supp}(f) \subset B_R$ and $\mathrm{supp}(g) \subset (0, T_0)$, where $B_R := \{x \in \mathbb{R}^3 |\ |x| < R\}$ for some $R$ and $T_0 > 0$. The inverse problem is to recover the source term $f$ from the Dirichlet boundary data $u$ measured on $\partial B_R \times (0,T)$ with $T > 0$ sufficiently large.

If we fix the parameter $\lambda$, it is in general impossible to uniquely recover a source term of the form $F(x,t)$, due to the presence of time-dependent non-radiating sources (see [4,5]). Motivated by [21], we suppose that the measurement data are given by a family of parameter-dependent functions $u_\lambda$ for all $\lambda \in (0, \Lambda^2)$ with $1 < \Lambda < \infty$. Here $u_\lambda$ denotes the unique solution to (1.1)-(1.2) corresponding to the parameter $\lambda$. The introduction of the parameter $\lambda$ serves as a theoretical tool to analyze the stability of the inverse source problem. In practice, while we cannot change the background medium, we can consider different scenarios where the wave speed varies naturally (e.g., different geological layers or different materials in engineering applications). The multi-speed measurements are used here to provide a more comprehensive understanding of the stability properties. We clarify in the paper that while the theoretical analysis is based on multi-speed measurements, the practical application should focus on scenarios where such variations occur naturally.

**(IP2):** Assume the source function $F$ has a compact support such that $\mathrm{supp}(F) \subset B_R \times (0, T_0)$. The inverse problem is to determine source term $F$ from the boundary observation data $\{u_\lambda(x,t) |\ x \in \partial B_R,\ t \in (0,T),\ \lambda \in (0, \Lambda^2)\}$. Here $T > T_0$ is sufficiently large.

In the last inverse problem, we suppose that the $x_3$-component of the source function is given.

**(IP3):** Assume $\lambda = 1$ and $F(x,t) = f(\tilde{x},t)g(x_3)$, where the function $f$ satisfies $\mathrm{supp}(f) \subset \widetilde{B}_{R_0} \times (0, T_0)$ and $g$ is supported in $(-R_0, R_0)$. Here $\tilde{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\widetilde{B}_{R_0} := \{\tilde{x} \in \mathbb{R}^2 |\ |\tilde{x}| < R_0\}$ The inverse problem is to determine the source term $f(\tilde{x},t)$ from the boundary observation data $u(x,t)$ on $\partial B_R \times (0,T)$ with $R > 0$, $T > 0$ sufficiently large.

The inverse source problems have many significant applications in scientific and engineering areas, as an important research subject in inverse scattering theory [26,27]. For instance, detection of submarines and non-destructive measurement of industrial objects can be regarded as recovery of acoustic sources from boundary measurements of the pressure. Other applications include biomedical imaging optical tomography [1,26] and geophysics. Consequently, inverse source problems have continuously attracted much attention by many researchers [2,3,5–11,19,20,26,39,41] and a great deal of mathematical and numerical results are available, especially for the time-dependent and time-harmonic acoustic waves.

Inverse source problems in the time domain are usually treated as hyperbolic systems by applying the Carleman estimates [33] and unique continuation theory. The approach of Carleman estimates can be used to recover both coefficients and source functions for hyperbolic equations; we refer to [12,32,41,42]. In the time-harmonic regime, it is well known that there is no uniqueness for inverse source problems with a single frequency due to the existence of non-radiating sources [4,22]. Computationally, a more serious issue is the lack of stability, i.e., a small variation of the data might lead to a huge error in the reconstruction. Hence it is crucial to study the stability of inverse source problems. The use of multiple-frequency data is an effective way to overcome non-uniqueness and has received a lot of attention in recent years. The paper [15] showed uniqueness and numerical results for the Helmholtz equation with multi-frequency data. The increasing stability results [5] handled more particular case and by quite different (direct spatial Fourier analysis) methods. The stability estimates consist of two parts: the Lipschitz type data discrepancy and the high frequency tail of the source functions. As the upper bound of frequencies increases, the latter decreases and thus becomes negligible. The increasing stability results revealed that ill-posedness of the inverse problems can be overcome by using multi-frequency data. A different method was proposed in [13] to derive increasing stability bounds for the Helmholtz equation in three dimensions, which involves the temporal Fourier transform together with sharp bounds of the analytic continuation at higher wave numbers. The authors of [13] firstly bridged the Helmholtz equation in the frequency-domain with the associated hyperbolic equations to get increasing stability results. These results were later generalized to the Helmholtz equation and Maxwell's equation in three dimensions (see [35] and [6]). We also refer to [21,43] for uniqueness results of time-dependent inverse source problems using the approach of Fourier-Laplace transform. The increasing stability of determining potentials for the Schödinger equation can be found in [28,31,38]. Recently, the papers [34,36] addressed the inverse problem of simultaneously recovering both the wave speed coefficient and an unknown initial condition (acting as the source) for the multidimensional wave equation from a single passive boundary measurement.

To the best of our knowledge, increasing stability results are only obtained for the reduced wave equation in the time-harmonic domain. It shows that the ill-posedness can be improved when the interval length of multi-frequencies or the energy at a fixed frequency increases. However, these results do not lead to increasing stability estimates for time-dependent wave equations, because the measurement data in the frequency domain cannot be bounded by the time-dependent measurement data in a trivial manner. Even the concept of increasing stability seems not available in the literature. The purpose of this paper is to attempt to study stability of inverse source problems for wave equations with explicit dependence on the interval length of given parameters (e.g., bandwith of the temporal component of the source function), which shows the same ill-posedness nature as the inverse time-harmonic problems. We are mostly motivated by the existing methods [30] in the frequency-domain and the uniqueness proof of [29,30] based on the Fourier transform. We firstly transform inverse source problems of the acoustic wave equation to those of the Helmholtz equation with multiple frequencies. Then we derive increasing stability estimate by using sharp bounds of analytic continuation given in [40], but keep the bound of the Dirichlet measurement data on the lateral boundary in the time domain. The existing approaches dealing with the Helmholtz equations has been adapted to handle inverse source problems for the wave equation in the time domain.

The rest of this paper is organized as follows. In Sections 2 and 3, we state main results and well-posedness of the direct problems. Section 4 is devoted to the increasing stability of the inverse problem (IP1). In Section 5, we investigate the second problem (IP2) for general source terms. The third inverse problem (IP3) will be treated in Section 6 and concluding remarks will be made in the final Section 7.

## 2. Main results

In this paper the source function $F$ is always required to be real-valued. The Fourier transform of $F$ with respect to time and spatial variables is defined as

$$\hat{F}(\xi,\omega) := (2\pi)^{-2}\int_{\mathbb{R}^4} F(x,t)e^{-i(\xi\cdot x+\omega t)}dxdt,$$

which implies that $\hat{F}(-\xi,-\omega)=\overline{\hat{F}}(\xi,\omega)$ for all $\xi\in\mathbb{R}^3$ and $\omega\in\mathbb{R}$.

First we show increasing stability for the time-dependent inverse problem (IP1). Let $F(x,t)=f(x)g(t)$. It is supposed that $f\in H^1(\mathbb{R}^3)$ and $g\in H^2(0,\infty)$, where $f$ is compactly supported in $B_R$ and $g$ is supported in $(0,T_0)$ for some $T_0>0$. The one-dimensional Fourier transform of $g$ with respect to the time variable $t$ takes the form:

$$\hat{g}(\omega)=(2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}} g(t)e^{-i\omega t}dt=(2\pi)^{-\frac{1}{2}}\int_0^{T_0} g(t)e^{-i\omega t}dt.$$

We suppose there exist numbers $b>1$ and $\delta>0$ such that

$$|\hat{g}(\omega)|\geq\delta>0 \quad \text{for all} \quad \omega\in(0,b). \tag{2.3}$$

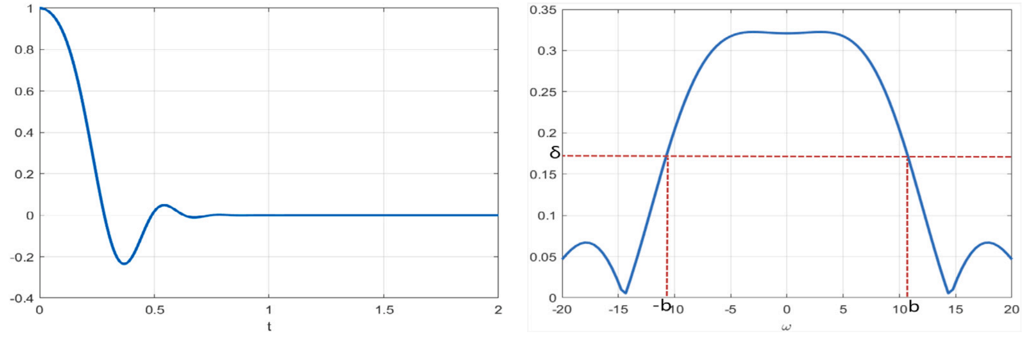
Physically, the parameter $b$ in (2.3) is associated with the bandwidth of the temporal signal $g(t)$. The condition (2.3) covers a large class of functions. For example, in many applications, the temporal function is usually given by the Ricker wavelet

$$g(t)=\cos[2(\pi f_p(t-t_0))^2]\,e^{-(\pi f_p(t-t_0))^2}$$

with the center frequency $f_p$ and the delay time $t_0$. In this paper, we suppose that $f_p=1$, $t_0=0$ and $g(t)$ is compactly supported. Thus choose

$$g(t)=\chi(t)\cos[2(\pi t)^2]\,e^{-(\pi t)^2},$$

where $\chi$ is a cut off function satisfying $\chi(t)\equiv 0$ when $|t|>T_0$. One can always find the parameters $b>0$ and $\delta>0$ such that (2.3) holds true (see Fig. 1). Since the source function is real




(a) Function $g(t)$ (b) Fourier transform $|\hat{g}(\omega)|$

Fig. 1. An example of $g(t)$ and its Fourier transform $|\hat{g}(\omega)|$.

valued, (2.3) holds true for all $\omega \in (-b, b)$. We remark that the interval $(0, b)$ in (2.3) can be replaced by $(\omega_0 - b, \omega_0 + b)$ for some $\omega_0 \geq 0$. In this paper we take $\omega_0 = 0$ for simplicity.

It is worth noting that the choice of the parameter $b$ given in (2.3) is not unique. Further, it is not related to the compact support $[0, T_0]$ of $g$, that is, when $T_0$ increases, $b$ may be not changed significantly. For example, if we choose the temporal function in the form

$$g(t) = \sin(at)\,\chi(t)$$

with the frequency $a > 0$, then the Fourier transform of $g$ is

$$\hat{g}(\omega) = -i(2\pi)^{-\frac{1}{2}}\left(\frac{\sin((a-\omega)T_0)}{a-\omega} - \frac{\sin((a+\omega)T_0)}{a+\omega}\right).$$

This expression is approximately the superposition of two sinc functions, with their main lobes concentrated at $\omega = \pm a$. Hence, the width of the interval $(0, b)$ remains almost the same when the parameter $T_0$ increases to infinity, which is shown in Fig. 2 with $a = \pi$, $T_0 = 2, 5, 200$.

Through this paper $C > 0$ is a generic constant which may vary from line to line. In the following theorem, we establish the increasing stability estimate of the $L^2$-norm of $f$ in terms of the parameter $b$ specified in (2.3).

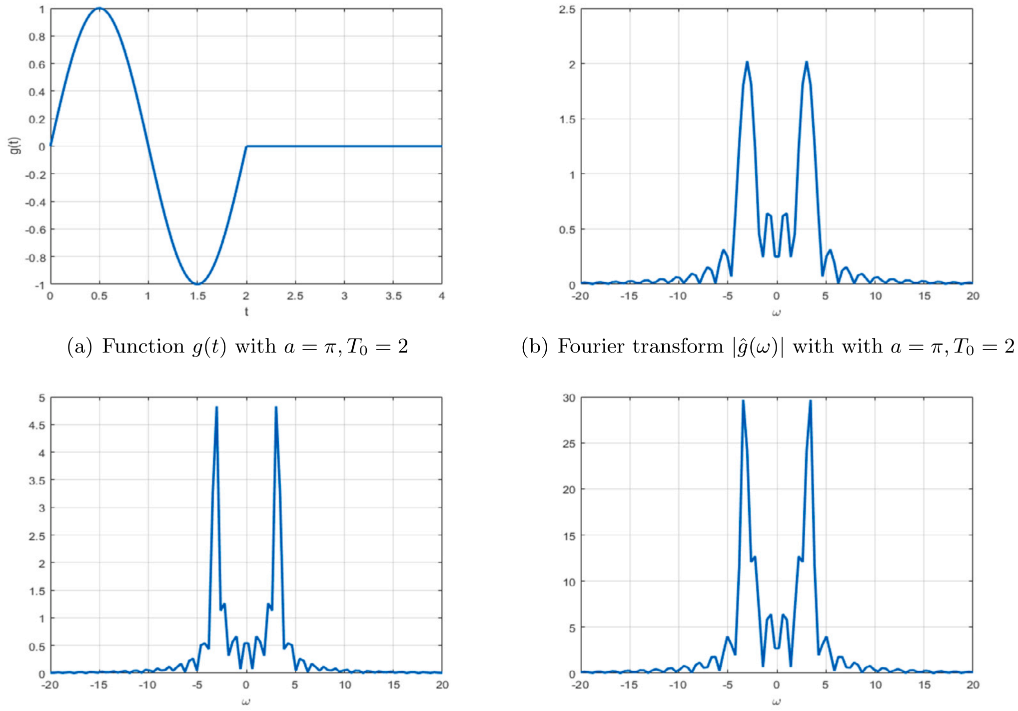
**Theorem 2.1.** *Let the condition (2.3) hold and let $T > 2R + T_0$. Assume that $g$ is given and $\|f\|_{H^1(\mathbb{R}^3)} \leq M$ where $M > 1$ is a constant. Then*

$$\|f\|^2_{L^2(\mathbb{R}^3)} \leq C\left(b^5\epsilon^2 + \frac{M^2}{b^{\frac{4}{3}}|\ln\epsilon|^{\frac{1}{2}}}\right), \tag{2.4}$$

*where $\epsilon = \|u\|_{H^2([0,T];H^{\frac{3}{2}}(\partial B_R))}$.*

**Remark 2.2.**

(1) There are two parts in the stability estimate (2.4). As the parameter $b$ increases, the latter decreases and thus becomes negligible. It is clear to conclude that the ill-posedness of the




(a) Function $g(t)$ with $a = \pi, T_0 = 2$

(b) Fourier transform $|\hat{g}(\omega)|$ with with $a = \pi, T_0 = 2$

(c) Fourier transform $|\hat{g}(\omega)|$ with $a = \pi, T_0 = 5$

(d) Fourier transform $|\hat{g}(\omega)|$ with $a = \pi, T_0 = 200$

Fig. 2. An example of $g(t)$ and its Fourier transform $|\hat{g}(\omega)|$.

inverse source problem decreases as the parameter $b$ increases. The estimate (2.4) is inconsistent with the increasing stability results of [35] in the frequency domain: the ill-posedness decreases when the width of the wave number interval increases.

(2) In Theorem 2.1, only Dirichlet boundary measurement data are needed. When the Cauchy data are available on the measurement boundary, one can obtain the same result (2.4) with a relaxed regularity assumption on the Dirichlet data, for example, $\epsilon = \|u\|_{L^2([0,T]\times\partial B_R)} + \|\frac{\partial u}{\partial \nu}\|_{L^2([0,T]\times\partial B_R)}$.

Next, we consider the second inverse problem (IP2). Our aim is to establish an increasing stability estimate of the $L^2$-norm of $F$ in terms of $\Lambda$.

**Theorem 2.3.** *Let $T > \frac{2R}{\lambda} + T_0$ and let $F \in H^3([0, T_0]; H^1(\mathbb{R}^3))$ be such that supp$(F)(x,t) \subset B_R \times (0, T_0)$. Assume $\|F\|_{H^3([0,T_0];H^1(\mathbb{R}^3))} \le M$ for some $M > 1$. Then there exist $C > 0$ and $\alpha \in (0, 1)$ such that*

$$\|F\|^2_{L^2(\mathbb{R}^4)} \le C\Big(\Lambda^{10}\epsilon^2 + \frac{M^2}{\Lambda|\ln\epsilon|^{\frac{1}{2}(1-\alpha)}}\Big), \tag{2.5}$$

*where* $\epsilon = \sup\limits_{0<\lambda<\Lambda^2} \|u_\lambda\|_{H^2([0,T];H^{\frac{3}{2}}(\partial B_R))}$.

Finally, we present the increasing stability for the third inverse problem (IP3). Suppose that $g$ is given and supported in $(-R_0, R_0)$ for some $0 < R_0 < R/\sqrt{2}$. The Fourier transform of $g(x_3)$ is given by

$$\hat{g}(\xi_3) = (2\pi)^{-\frac{1}{2}} \int\limits_{\mathbb{R}} g(x_3) e^{-i\xi_3 \cdot x_3} dx_3.$$

We suppose

$$|\hat{g}(\xi_3)| \geq \delta > 0 \quad \text{for all} \quad \xi_3 \in (-b, b), \tag{2.6}$$

where $b > 1$.

**Theorem 2.4.** *Let $T > 2R + T_0$ and let $f(\tilde{x}, t) \in H^2([0, T_0]; H^1(\mathbb{R}^2))$ be such that supp $f(\tilde{x}, t) \subset \widetilde{B}_{R_0} \times (0, T_0)$. Assume $\|f\|_{H^1(\widetilde{B}_{R_0} \times (0,T_0))} \leq M$ for some $M > 1$. Then there exist $C > 0$ and $\alpha \in (0, 1)$ such that*

$$\|f\|^2_{L^2(\mathbb{R}^3)} \leq C\Big(b^5\epsilon^2 + b^3\, e^{2b(1-\alpha)}(1+b)^{2\alpha}\epsilon^{2\alpha} + \frac{M^2}{b^{\frac{4}{3}}|\alpha \ln \epsilon|^{\frac{1}{2}}}\Big), \tag{2.7}$$

*where $\epsilon = \|u\|_{H^2([0,T]; H^{\frac{3}{2}}(\partial B_R))}$.*

## 3. Preliminaries

The increasing stability results of this paper are crucially dependent on the stability estimates of the forward problems. Although partial results of this section are well known, we still present them for the readers' convenience and also to make this paper self-contained. Consider the mathematical model of the time-dependent wave propagation problem caused by the source term $f(x)g(t)$,

$$\begin{cases} \partial_t^2 u - \Delta u = f(x)g(t), & (x,t) \in \mathbb{R}^3 \times (0, +\infty), \\ u(x, 0) = \partial_t u(x, 0) = 0, & x \in \mathbb{R}^3. \end{cases} \tag{3.8}$$

Assume $f \in L^2(\mathbb{R}^3)$ is compactly supported in $B_R$ and $g \in L^2(0, \infty)$ is supported in $(0, T)$. Then the problem (3.8) admits a unique solution

$$u(x, t) \in \mathcal{C}([0, +\infty); H^1(\mathbb{R}^3)) \cap \mathcal{C}^1([0, +\infty); L^2(\mathbb{R}^3)).$$

The well-posed result can be easily proved by using the elliptic regularity properties of the Laplace operator (see [14,16–18,23,37]). Now we state the regularity of the solution for the initial boundary value problem (3.8). Below we suppose that $T > 0$ is sufficiently large.

**Lemma 3.1.** *For $p > 0$, let $f(x) \in H^p(\mathbb{R}^3)$ be supported in $B_R$ and $g(t) \in L^2([0, T])$. Then the problem (3.8) admits a unique solution $u(x, t) \in \mathcal{C}([0, T]; H^{p+1}(\mathbb{R}^3)) \cap H^2([0, T]; H^{p-1}(\mathbb{R}^3))$ satisfying*

$$\|u\|_{\mathcal{C}([0,T];H^{p+1}(\mathbb{R}^3))} + \|u\|_{H^2([0,T];H^{p-1}(\mathbb{R}^3))} \leqslant C\|g\|_{L^2[0,T]}\|f\|_{H^p(\mathbb{R}^3)}, \tag{3.9}$$

*with the positive constant $C$ depending on $R$ and $T$.*

**Proof.** Denote the Fourier transform of $u(x,t)$ with respect to the spatial variables as follows:

$$\hat{u}(\xi,t) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} u(x,t)e^{-i\xi\cdot x}dx, \quad \xi \in \mathbb{R}^3.$$

Since $\widehat{\Delta u}(x,t) = -|\xi|^2\hat{u}(\xi,t)$, the function $\hat{u}(\xi,t)$ solves the ordinary differential equation

$$\begin{cases} \partial_t^2\hat{u}(\xi,t) + |\xi|^2\hat{u}(\xi,t) = \hat{f}(\xi)g(t), & (\xi,t) \in \mathbb{R}^3 \times (0,T), \\ \hat{u}(\xi,0) = 0, \quad \partial_t\hat{u}(\xi,0) = 0, & \xi \in \mathbb{R}^3. \end{cases} \tag{3.10}$$

By Duhamel's principle, it is easy to check that the unique solution to (3.10) takes the form

$$\hat{u}(\xi,t) = \int_0^t H(\xi,t-s)g(s)\,\mathrm{d}s,$$

where the function $H$ is defined as

$$H(\xi,t-s) = |\xi|^{-1}\sin(|\xi|(t-s))\,\widehat{f}(\xi), \quad \text{for all} \quad \xi \in \mathbb{R}^3,\ t,s \in [0,T].$$

Since $f$ is compactly supported in $B_R$, we obtain

$$\begin{aligned} \|H(\cdot,t-s)\|^2_{L^2(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^{-2}\sin^2(|\xi|(t-s))|\widehat{f}(\xi)|^2\,\mathrm{d}\xi \\ &\leqslant \int_{B_1} |\xi|^{-2}\sin^2(|\xi|(t-s))|\widehat{f}(\xi)|^2\,\mathrm{d}\xi + \int_{\mathbb{R}^3\setminus B_1} |\xi|^{-2}|\widehat{f}(\xi)|^2\,\mathrm{d}\xi \\ &\leqslant C|B_1|^2\|f\|^2_{L^2(\mathbb{R}^3)} + ||f||^2_{L^2(\mathbb{R}^3)} \\ &\leqslant (1 + C|B_1|^2)\|f\|^2_{L^2(\mathbb{R}^3)} \end{aligned} \tag{3.11}$$

for some positive constant $C$, where $B_1 := \{x \in \mathbb{R}^3 |\ |x| < 1\}$. In the same way, it follows for $p > 0$ that

$$\begin{aligned} \|(1+|\xi|^2)^{\frac{p+1}{2}}H(\cdot,t-s)\|^2_{L^2(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} (1+|\xi|^2)^{p+1}|\xi|^{-2}\sin^2(|\xi|(t-s))|\widehat{f}(\xi)|^2\,\mathrm{d}\xi \\ &\leqslant \int_{B_1} (1+|\xi|^2)^{p+1}|\xi|^{-2}\sin^2(|\xi|(t-s))|\widehat{f}(\xi)|^2\,\mathrm{d}\xi \end{aligned}$$

$$
\begin{aligned}
&\quad + \int\limits_{\mathbb{R}^3\setminus B_1} (1+|\xi|^2)^{p+1}|\xi|^{-2}|\widehat{f}(\xi)|^2\,\mathrm{d}\xi \\
&\leqslant C\int\limits_{\mathbb{R}^3}(1+|\xi|^2)^p|\widehat{f}(\xi)|^2\,\mathrm{d}\xi \\
&\quad + 2\int\limits_{\mathbb{R}^3\setminus B_1} (1+|\xi|^2)^{p+1}\frac{1}{2|\xi|^2}|\widehat{f}(\xi)|^2\,\mathrm{d}\xi \\
&\leqslant C\int\limits_{\mathbb{R}^3}(1+|\xi|^2)^p|\widehat{f}(\xi)|^2\,\mathrm{d}\xi + 2\int\limits_{\mathbb{R}^3}(1+|\xi|^2)^p|\widehat{f}(\xi)|^2\,\mathrm{d}\xi \\
&\leqslant (C+2)\|f\|^2_{H^p(\mathbb{R}^3)}.
\end{aligned} \tag{3.12}
$$

In view of the estimates (3.11) and (3.12), it's easy to deduce that $u(x,t)\in\mathcal{C}([0,T];H^{p+1}(\mathbb{R}^3))$. Next we consider its derivative with respect to time. For $\xi\in\mathbb{R}^3$ we have

$$
\partial_t\hat{u}(\xi,t) = \int\limits_0^t \partial_t H(\xi,t-s)g(s)\,\mathrm{d}s = \int\limits_0^t \cos(|\xi|(t-s))\,\hat{f}(\xi)g(s)\,\mathrm{d}s
$$

and

$$
\begin{aligned}
\partial_t^2\hat{u}(\xi,t) &= g(t)\hat{f}(\xi) + \int\limits_0^t \partial_t^2 H(\xi,t-s)g(s)\,\mathrm{d}s \\
&= g(t)\hat{f}(\xi) + \int\limits_0^t -|\xi|\sin(|\xi|(t-s))\,\hat{f}(\xi)g(s)\,\mathrm{d}s.
\end{aligned}
$$

Combining this with (3.12), we find

$$
\|(1+|\xi|^2)^{\frac{p}{2}}\partial_t H(\xi,t-s)\|^2_{L^2(\mathbb{R}^3)} \leqslant C\|f\|^2_{H^p(\mathbb{R}^3)}
$$

and

$$
\|(1+|\xi|^2)^{\frac{p-1}{2}}\partial_t^2 H(\xi,t-s)\|^2_{L^2(\mathbb{R}^3)} \leqslant C\|f\|^2_{H^p(\mathbb{R}^3)},
$$

which proves $u(x,t)\in H^2([0,T];H^{p-1}(\mathbb{R}^3))$. The estimate (3.9) is also easy to get from the previous estimates. □

In Theorems 2.1-2.4, we obtain increasing stability results by using only the Dirichlet boundary data rather than the Cauchy data. This is due to the fact that our inverse problems are all formulated in the unbounded domain $|x|>R$ and the Neumann data on $\partial B_R\times(0,T)$ can be controlled by the Dirichlet data on $\partial B_R\times(0,T)$. To prove this rigorously, we need to impose higher regularity assumptions on $f$ and $g$.

**Lemma 3.2.** *Let $f \in H^1(\mathbb{R}^3)$ and $g \in H^2([0,T])$. Then the problem (3.8) admits a unique solution $u \in \mathcal{C}^2([0,T]; H^2(\mathbb{R}^3)) \cap H^2([0,T]; L^2(\mathbb{R}^3))$ satisfying the estimate*

$$\|\partial_\nu u\|_{L^2([0,T]; H^{\frac{1}{2}}(\partial B_R))} \leqslant C\|u\|_{H^2([0,T]; H^{\frac{3}{2}}(\partial B_R))} \tag{3.13}$$

*with positive constant $C$ depending on $T$ and $R$.*

**Proof.** Applying Lemma 3.1 with $p = 1$, we know

$$\begin{aligned} u(x,t) &\in \mathcal{C}^2([0,T]; H^2(\mathbb{R}^3)) \cap H^4([0,T]; L^2(\mathbb{R}^3)), \\ h(x,t) := u|_{\partial B_R \times [0,T]} &\in \mathcal{C}^2([0,T]; H^{\frac{3}{2}}(\partial B_R)). \end{aligned} \tag{3.14}$$

Therefore, the restriction of $u(x,t)$ to $(\mathbb{R}^3 \backslash \overline{B_R}) \times [0,T]$ solves the initial boundary value problem

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) = 0, & (x,t) \in (\mathbb{R}^3 \backslash \overline{B_R}) \times (0,T), \\ u(x,0) = \partial_t u(x,0) = 0, & x \in \mathbb{R}^3 \backslash \overline{B_R}, \\ u(x,t) = h(x,t), & (x,t) \in \partial B_R \times (0,T). \end{cases} \tag{3.15}$$

Combining a classical lifting result with the fact that $h(x,0) = h_t(x,0) = 0$, we introduce a function $H(x,t) \in \mathcal{C}^2([0,T]; H^2(\mathbb{R}^3 \backslash B_R))$ such that $H|_{\partial B_R \times [0,T]} = h$, $H(x,0) = \partial_t H(x,0) = 0$ and

$$\|H\|_{H^2([0,T]; H^2(\mathbb{R}^3 \backslash B_R))} \leqslant C\|h\|_{H^2([0,T]; H^{\frac{3}{2}}(\partial B_R))}, \tag{3.16}$$

where $C > 0$ depends on $R$ and $T$. Therefore, we can split $u$ to $u = H + V$ on $\mathbb{R}^3 \backslash B_R \times (0,T)$. Here $V$ solves

$$\begin{cases} \partial_t^2 V(x,t) - \Delta V(x,t) = -(\partial_t^2 H(x,t) - \Delta H(x,t)) := G, & (x,t) \in \mathbb{R}^3 \backslash \overline{B_R} \times (0,T), \\ V(x,0) = \partial_t V(x,0) = 0, & x \in \mathbb{R}^3 \backslash \overline{B_R}, \\ V(x,t) = 0, & (x,t) \in \partial B_R \times (0,T). \end{cases}$$

Using the fact that $G \in L^2([0,T]; L^2(\mathbb{R}^3 \backslash B_R)$, in a similar way to Lemma 3.1, one can prove that $V \in \mathcal{C}([0,T]; H^2(\mathbb{R}^3))$ satisfies the estimate

$$\|V\|_{L^2([0,T]; H^2(\mathbb{R}^3))} \leq C\|G\|_{L^2([0,T]; L^2(\mathbb{R}^3 \backslash \overline{B_R}))} \leq C\|H\|_{H^2([0,T]; H^2(\mathbb{R}^3 \backslash \overline{B_R}))}.$$

Combining this with (3.16), we deduce that

$$\|u\|_{L^2([0,T]; H^2(\mathbb{R}^3 \backslash \overline{B_R})} \leqslant C\|h\|_{H^2([0,T]; H^{\frac{3}{2}}(\partial B_R))}$$

and using the continuity of the trace map, we obtain

$$\|\partial_\nu u\|_{L^2([0,T]; H^{\frac{1}{2}}(\partial B_R))} \leqslant C\|u\|_{L^2([0,T]; H^2(\mathbb{R}^3 \backslash \overline{B_R}))}.$$

Combining the last two estimates with the definition of $h$ given by (3.14), we finally obtain (3.13). □

Below we state a stability estimate for analytic continuation problems, which can be seen in [9,40].

**Proposition 3.3.** *Let $\mathcal{O}$ be a non empty open set of the unit ball $B_1 \subset \mathbb{R}^d$, $d \geq 2$, and let $G$ be an analytic function in $B_2$, that satisfies*

$$\|\partial^\gamma G\|_{L^\infty(B_2)} \leq M_0\, |\gamma|!\, \eta^{-|\gamma|}, \quad \forall \gamma \in (\mathbb{N} \cup \{0\})^d,$$

*for some $M_0 > 0$ and $\eta > 0$. Then, we have*

$$\|G\|_{L^\infty(B_1)} \leq N\, M_0^{1-\mu}\, \|G\|^{\mu}_{L^\infty(\mathcal{O})},$$

*where $\mu \in (0,1)$ depends on $d$, $\eta$, $|\mathcal{O}|$ and $N = N(\eta) > 0$.*

## 4. Proof of Theorem 2.1

In this section, we discuss increasing stability of the inverse problem (IP1). Firstly we introduce the time-dependent test function

$$w(x,t) := e^{-i\xi\cdot x - i\omega t},$$

where $\xi \in \mathbb{R}^3$ and $\omega \in \mathbb{R}$ satisfy $|\xi|^2 = \omega^2$. It is obvious that $w$ satisfies the homogeneous acoustic wave equation

$$\partial_t^2 w - \Delta w = 0 \qquad \text{in } \mathbb{R}^3 \times (0, +\infty). \tag{4.17}$$

Then, multiplying $w$ on both sides of the equation (1.1) and integrating over $B_R \times (0,T)$, we obtain

$$\int_0^T \int_{B_R} (\partial_t^2 u - \Delta u) w \,\mathrm{d}x\mathrm{d}t = \int_0^T \int_{B_R} f(x) g(t) w(x,t)\,\mathrm{d}x\mathrm{d}t. \tag{4.18}$$

Using (4.17) and integrating by parts, one deduces from the left hand side of (4.18) that

$$\begin{aligned}
&\int_0^T \int_{B_R} (\partial_t^2 u(x,t) - \Delta u(x,t)) w(x,t)\,\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{B_R} \left(\partial_t^2 u(x,t) w(x,t) - u(x,t)\partial_t^2 w(x,t)\right) \mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T\int_{\partial B_R}\big(u(x,t)\frac{\partial w(x,t)}{\partial\nu}-w(x,t)\frac{\partial u(x,t)}{\partial\nu}\big)\,\mathrm{d}s(x)\mathrm{d}t\\
=&\int_{B_R}\big(\partial_t u(x,t)w(x,t)-u(x,t)\partial_t w(x,t)\big)\big|_0^T\,\mathrm{d}x\\
&+\int_0^T\int_{\partial B_R}\big(u(x,t)\frac{\partial w(x,t)}{\partial\nu}-w(x,t)\frac{\partial u(x,t)}{\partial\nu}\big)\,\mathrm{d}s(x)\mathrm{d}t\\
=&\int_0^T\int_{\partial B_R}\big(u(x,t)\frac{\partial w(x,t)}{\partial\nu}-w(x,t)\frac{\partial u(x,t)}{\partial\nu}\big)\,\mathrm{d}s(x)\mathrm{d}t.
\end{aligned}
$$

Note that, in the last step we have used the fact that $u(x,t)=0$ when $|x|<R$ and $t>T_0+2R$, which follows straightforwardly from Huygens' principle (see [10, Lemma 2.1]). This implies $u(x,T)=\partial_t u(x,T)=0$ for $x\in B_R$ and $T>T_0+2R$. Hence, the integral over $B_R$ on the left hand side of the previous identity vanishes. Recalling the estimate in Lemma 3.2 with Sobolev's embedding theorems, we bound the left hand side of (4.18) by

$$
\begin{aligned}
&\Big|\int_0^T\int_{B_R}(\partial_t^2 u(x,t)-\Delta u(x,t))w(x,t)\,\mathrm{d}x\mathrm{d}t\Big|\\
=&\Big|\int_0^T\int_{\partial B_R}(u(x,t)\frac{\partial w(x,t)}{\partial\nu}-w(x,t)\frac{\partial u(x,t)}{\partial\nu})\,\mathrm{d}s(x)\mathrm{d}t\Big|\\
\le&\|u\|_{L^2([0,T];L^2(\partial B_R))}\|\frac{\partial w}{\partial\nu}\|_{L^2([0,T];L^2(\partial B_R))}+\|\frac{\partial u}{\partial\nu}\|_{L^2([0,T];L^2(\partial B_R))}\|w\|_{L^2([0,T];L^2(\partial B_R))}\\
\le&C(\|u\|_{L^2([0,T];L^2(\partial B_R))}\|w\|_{L^2([0,T];H^2(B_R))}+\|u\|_{H^2([0,T];H^{\frac{3}{2}}(\partial B_R))}\|w\|_{L^2([0,T];L^2(\partial B_R))})\\
\le&C\|u\|_{H^2([0,T];H^{\frac{3}{2}}(\partial B_R))}\|w\|_{L^2([0,T];H^2(B_R))}.
\end{aligned}
$$

By the definition of $w$, one can check that

$$
\|w\|_{L^2([0,T];H^2(B_R))}\le C\,(1+|\xi|)\qquad\text{for all}\quad |\xi|^2=\omega^2.
$$

Hence, using the assumption about $g$ and the fact that $f$ is supported in $B_R$, we derive from (4.18) together with the previous two relations that

$$
|\hat{f}(\xi)\hat{g}(\omega)|=\Big|(2\pi)^{-2}\int_0^T\int_{\mathbb{R}^3}f(x)g(t)e^{-i\xi\cdot x-i\omega t}\,\mathrm{d}x\mathrm{d}t\Big|
$$

$$
\begin{aligned}
&= \left| (2\pi)^{-2} \int_0^T \int_{B_R} f(x) g(t) w(x,t)\, \mathrm{d}x \mathrm{d}t \right| \\
&\le C(1+|\xi|) \|u\|_{H^2([0,T];H^{\frac{3}{2}}(\partial B_R))}.
\end{aligned}
$$

In view of the assumption (2.3), one obtains for $\omega \in (0, b)$ and $|\xi|^2 = \omega^2$ that

$$
|\hat{f}(\xi)| \le C \frac{(1+|\xi|)\|u\|_{H^2([0,T],H^{\frac{3}{2}}(\partial B_R))}}{|\hat{g}(\omega)|} \le C\,\delta^{-1}(1+|\xi|)\|u\|_{H^2([0,T];H^{\frac{3}{2}}(\partial B_R))}. \tag{4.19}
$$

We note that (4.19) gives an estimate of $\hat{f}(\xi)$ over the domain $E := B(0,b) = \{\xi \in \mathbb{R}^3 |\ |\xi| < b\}$, that is,

$$
\|\hat{f}\|_{L^\infty(E)} \le C\,\delta^{-1}(1+b)\,\epsilon, \tag{4.20}
$$

where $\epsilon = \|u\|_{H^2([0,T];H^{\frac{3}{2}}(\partial B_R))}$ represents the measurement data on $\partial B_R \times (0,T)$. Applying the Parseval's identity, we have

$$
\|f\|^2_{L^2(\mathbb{R}^3)} = \|\hat{f}\|^2_{L^2(\mathbb{R}^3)} = I(k) + \int_{|\xi|>k} |\hat{f}(\xi)|^2\, \mathrm{d}\xi,
$$

where

$$
I(k) := \int_{|\xi|\le k} |\hat{f}(\xi)|^2\, \mathrm{d}\xi = \int_0^k \int_{\mathbb{S}^2} |\hat{f}(l\theta)|^2 l^2 \mathrm{d}\theta \mathrm{d}l. \tag{4.21}
$$

Since $\hat{f}$ is an entire analytic function of $\xi$, $I(k)$ is an entire analytic function of $k = k_1 + ik_2$ ($k_1, k_2 \in \mathbb{R}$) and the following estimate holds.

**Lemma 4.1.** *Let $f \in L^2(\mathbb{R}^3)$ and* supp $f \subset B_R$. *Then*

$$
|I(k)| \le (\frac{8\pi}{9})\, R^3\, |k|^3 e^{2R|k_2|} \|f\|^2_{L^2(\mathbb{R}^3)}, \qquad k = k_1 + ik_2 \in \mathbb{C}. \tag{4.22}
$$

**Proof.** Set $l = ks$ for $s \in (0,1)$. Then it is easy to get

$$
I(k) = \int_0^k \int_{\mathbb{S}^2} |\hat{f}(l\theta)|^2 l^2 \mathrm{d}\theta \mathrm{d}l = \int_0^1 \int_{\mathbb{S}^2} |\hat{f}(ks\theta)|^2 k^3 s^2 \mathrm{d}\theta \mathrm{d}s.
$$

Noting the elementary inequality $|e^{2k_2 s\theta\cdot x}| \le e^{2R|k_2|}$ for all $x \in B_R$ and $\theta \in \mathbb{S}^2$, we have

$$
\begin{aligned}
|I(k)| &= \left| \int_0^1 \int_{\mathbb{S}^2} \frac{1}{\sqrt{2\pi}} \left| \int_{B_R} f(x) e^{-iks\theta\cdot x} \mathrm{d}x \right|^2 k^3 s^2 \mathrm{d}\theta \mathrm{d}s \right| \\
&\le \frac{2}{3} R^3 \int_0^1 \int_{\mathbb{S}^2} |k|^3 s^2 \left( \int_{B_R} |f(x)|^2 |e^{2k_2 s\theta\cdot x}| \,\mathrm{d}x \right) \mathrm{d}\theta \mathrm{d}s \\
&\le (\frac{8\pi}{9})\, R^3\, |k|^3 e^{2R|k_2|} \|f\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

This completes the Lemma 4.1. □

The following Lemma is essential to show the relation between $I(k)$ for $k \in (b, \infty)$ with $I(b)$. Its proof can be found in [13].

**Lemma 4.2.** *Let $J(z)$ be an analytic function in $S = \{z = x + iy \in \mathbb{C} : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}$ and continuous in $\overline{S}$ satisfying*

$$
\begin{cases}
|J(z)| \le \epsilon, & z \in (0,\ L], \\
|J(z)| \le V, & z \in S, \\
|J(0)| = 0.
\end{cases}
$$

*Then there exists a function $\mu(z)$ satisfying*

$$
\begin{cases}
\mu(z) \ge \frac{1}{2}, & z \in (L,\ 2^{\frac{1}{4}} L), \\
\mu(z) \ge \frac{1}{\pi}((\frac{z}{L})^4 - 1)^{-\frac{1}{2}}, & z \in (2^{\frac{1}{4}} L,\ \infty)
\end{cases}
$$

*such that*

$$
|J(z)| \le V \epsilon^{\mu(z)}, \quad \forall z \in (L,\ \infty).
$$

Let the sector $S \subset \mathbb{C}$ be given as in Lemma 4.2. Now, it follows from Lemma 4.1 that

$$
|I(k) e^{-(2R+1)k}| \le C M^2 \quad \text{for all} \quad k \in S.
$$

Recalling from the a priori estimate (4.20), we obtain

$$
I(k) = \int_{|\xi| \le k} |\hat{f}(\xi)|^2 d\xi \le C\, k^3\, (1+k)^2\, \epsilon^2, \quad k \in (0, b].
$$

Hence

$$
|I(k) e^{-(2R+1)k}| \le C \epsilon^2, \;\; k \in (0, b].
$$

Then applying Lemma 4.2 with $L = b$ to the function $J(k) := I(k)e^{-(2R+1)k}$, we know that there exists a function $\mu(k)$ satisfying

$$\begin{cases} \mu(k) \geq \frac{1}{2}, & k \in (b,\ 2^{\frac{1}{4}}b), \\ \mu(k) \geq \frac{1}{\pi}((\frac{k}{b})^4 - 1)^{-\frac{1}{2}}, & k \in (2^{\frac{1}{4}}b,\ \infty) \end{cases} \tag{4.23}$$

such that

$$|I(k)e^{-(2R+1)k}| \leq CM^2\epsilon^{2\mu} \quad \text{for all} \quad k \in (b, \infty).$$

**Proof of Theorem 2.1.** We assume that $\epsilon < e^{-1}$. The estimate is obvious if otherwise. Let

$$k = \begin{cases} \frac{1}{((2R+3)\pi)^{\frac{1}{3}}} b^{\frac{2}{3}} |\ln\epsilon|^{\frac{1}{4}}, & \text{if} \quad 2^{\frac{1}{4}}((2R+3)\pi)^{\frac{1}{3}} b^{\frac{1}{3}} < |\ln\epsilon|^{\frac{1}{4}}, \\ b, & \text{if} \quad |\ln\epsilon|^{\frac{1}{4}} \leq 2^{\frac{1}{4}}((2R+3)\pi)^{\frac{1}{3}} b^{\frac{1}{3}}. \end{cases}$$

Case (i): $2^{\frac{1}{4}}((2R+3)\pi)^{\frac{1}{3}} b^{\frac{1}{3}} < |\ln\epsilon|^{\frac{1}{4}}$. Then we have

$$\begin{aligned} |I(k)| &\leq CM^2\epsilon^{2\mu}e^{(2R+1)k} \\ &= CM^2 e^{(2R+1)k - 2\mu(k)|\ln\epsilon|} \\ &\leq CM^2 e^{\frac{(2R+3)}{((2R+3)\pi)^{\frac{1}{3}}} b^{\frac{2}{3}} |\ln\epsilon|^{\frac{1}{4}} - \frac{2|\ln\epsilon|}{\pi}(\frac{b}{k})^2} \\ &= CM^2 e^{-2(\frac{(2R+3)^2}{\pi})^{\frac{1}{3}} b^{\frac{2}{3}} |\ln\epsilon|^{\frac{1}{2}}(1 - \frac{1}{2}|\ln\epsilon|^{-\frac{1}{4}})}. \end{aligned}$$

Noting that $|\ln\epsilon|^{-\frac{1}{4}} < 1$ and $\left(\frac{(2R+3)^2}{\pi}\right)^{\frac{1}{3}} > 1$, we have

$$|I(k)| \leq CM^2 e^{-b^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{2}}}.$$

Using the inequality $e^{-t} \leq \frac{6!}{t^6}$ for $t > 0$, we get

$$|I(k)| \leq CM^2 \frac{1}{b^4|\ln\epsilon|^3}. \tag{4.24}$$

Since $b^4|\ln\epsilon|^3 \geq b^{\frac{4}{3}}|\ln\epsilon|^{\frac{1}{2}}$ when $b > 1$ and $|\ln\epsilon| > 1$, we use

$$\int_{|\xi|>k} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \leq \frac{1}{k^2} \int_{|\xi|>k} |\widehat{\nabla f}(\xi)|^2 \,\mathrm{d}\xi \leq \frac{M^2}{k^2}.$$

Hence

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{R}^3)} &= I(k) + \int\limits_{|\xi|>k} |\widehat{f}(\xi)|^2\,\mathrm{d}\xi \\ &\le I(k) + \frac{M^2}{k^2} \\ &\le C\Big(\frac{M^2}{b^4|\ln\epsilon|^3} + \frac{M^2}{b^{\frac{4}{3}}|\ln\epsilon|^{\frac{1}{2}}}\Big) \\ &\le \frac{CM^2}{b^{\frac{4}{3}}|\ln\epsilon|^{\frac{1}{2}}}.\end{aligned} \tag{4.25}$$

Case (ii): $|\ln\epsilon|^{\frac{1}{4}} \le 2^{\frac{1}{4}}((2R+3)\pi)^{\frac{1}{3}}b^{\frac{1}{3}}$. In this case we have $k = b$ by the choice of $k$ and $|I(b)| \le Cb^5\epsilon^2$. Hence,

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{R}^3)} &= I(b) + \int\limits_{|\xi|>b} |\widehat{f}(\xi)|^2\,\mathrm{d}\xi \\ &\le C\Big(b^5\epsilon^2 + \frac{M^2}{b^2}\Big) \\ &\le C\Big(b^5\epsilon^2 + \frac{M^2}{b^{\frac{4}{3}}|\ln\epsilon|^{\frac{1}{2}}}\Big).\end{aligned} \tag{4.26}$$

Combining (4.25) and (4.26), we finally get

$$\|f\|^2_{L^2(\mathbb{R}^3)} \le C\Big(b^5\epsilon^2 + \frac{M^2}{b^{\frac{4}{3}}|\ln\epsilon|^{\frac{1}{2}}}\Big). \qquad \square$$

## 5. Proof of Theorem 2.3

In this section, we consider the following initial value problem for the wave equation

$$\begin{cases}\partial_t^2 u_\lambda(x,t) - \lambda\Delta u_\lambda(x,t) = F(x,t), & (x,t)\in\mathbb{R}^3\times(0,\infty),\\ u(x,0)=0, \quad \partial_t u(x,0)=0, & x\in\mathbb{R}^3.\end{cases}$$

Suppose $\mathrm{supp}F(x,t) \subset B_R \times (0, T_0)$. Our aim is to recover the compacted supported function $F$ from the data $\{u_\lambda(x,t)|\ x\in\partial B_R,\ t\in(0,T), 0<\lambda<\Lambda^2\}$. Physically, such kind of the measurement data can be obtained by changing the background medium artificially and locally, for the purpose of recovering a time-dependent source term which might be non-radiating for a fixed parameter $\lambda$.

As done in Section 4 we introduce the test function

$$w(x,t) = e^{-i(\xi\cdot x+\omega t)}, \qquad \omega^2 - \lambda|\xi|^2 = 0.$$

One can check that $\partial_t^2 w - \lambda\Delta w = 0$. In this case again using the strong Huygens' principle gives that $u_\lambda(x,t) = 0$ when $|x| < R$ and $t > T_0 + \frac{2R}{\lambda}$ (see [10, Lemma 2.1]).

Multiplying $w$ on both sides of the wave equation (1.1) and integrating over $B_R \times (0,T)$, we obtain

$$\int_0^T \int_{B_R} F(x,t)e^{-i(\xi\cdot x+\omega t)}dxdt = \int_0^T \int_{B_R} \big(\partial_t^2 u_\lambda(x,t) - \lambda\Delta u_\lambda(x,t)\big)w(x,t)\,\mathrm{d}x\mathrm{d}t$$
$$= \lambda \int_0^T \int_{\partial B_R} \Big(u_\lambda(x,t)\frac{\partial w(x,t)}{\partial \nu} - w(x,t)\frac{\partial u_\lambda(x,t)}{\partial \nu}\Big)\,\mathrm{d}s(x)\mathrm{d}t.$$

Define

$$\hat{F}(\xi,\omega) = (2\pi)^{-2}\int_{\mathbb{R}^4} F(x,t)e^{-i(\xi\cdot x+\omega t)}dxdt,$$

with $(\xi,\omega) = (\xi_1,\xi_2,\xi_3,\omega) \in \mathbb{R}^4$. Since supp $F(x,t) \subset B_R \times (0,T_0)$, we have

$$(2\pi)^2\hat{F}(\xi,\omega) = \int_0^T \int_{B_R} F(x,t)e^{-i(\xi\cdot x+\omega t)}dxdt$$
$$= \lambda \int_0^T \int_{\partial B_R} \Big(u_\lambda(x,t)\frac{\partial w(x,t)}{\partial \nu} - w(x,t)\frac{\partial u_\lambda(x,t)}{\partial \nu}\Big)\,\mathrm{d}s(x)\mathrm{d}t. \tag{5.1}$$

Using the fact that $F(x,t)$ has compact support and arguing similarly to the proof of Lemma 3.2, we deduce

$$\|\partial_\nu u_\lambda\|_{L^2([0,T];H^{\frac{1}{2}}(\partial B_R))} \leqslant C\|u_\lambda\|_{H^2([0,T];H^{\frac{3}{2}}(\partial B_R))}. \tag{5.2}$$

Consider the set (see Fig. 3)

$$E(s) = \{(\xi,\omega)\in\mathbb{R}^4 |\ \omega^2 - \lambda|\xi|^2 = 0,\ |\xi|<s,\ 0<\lambda<s^2\}.$$

Combining (5.1) and (5.2), we have

$$|\hat{F}(\xi,\omega)| \le C\lambda(1+|\xi|)\|u_\lambda\|_{H^2([0,T];H^{\frac{3}{2}}(\partial B_R))} \le C\Lambda^2(1+\Lambda)\epsilon, \quad (\xi,\omega)\in E(\Lambda), \tag{5.3}$$

where $\epsilon = \sup\limits_{0<\lambda<\Lambda^2} \|u_\lambda\|_{H^2([0,T];H^{\frac{3}{2}}(\partial B_R))}$. Denote $I(s) := \int_{E(s)} |\hat{F}(\xi,\omega)|^2 d\xi\, d\omega$. Then

$$|I(s)| \le C\,|E(s)|s^4\,(1+s)^2\,\epsilon^2 \quad \text{for all} \quad s\in[0,\Lambda]. \tag{5.4}$$

Using the polar coordinates $\xi = r\hat{\xi} = r(\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$ for $0\le\theta\le 2\pi$ and $0\le\varphi\le\pi$, we deduce that

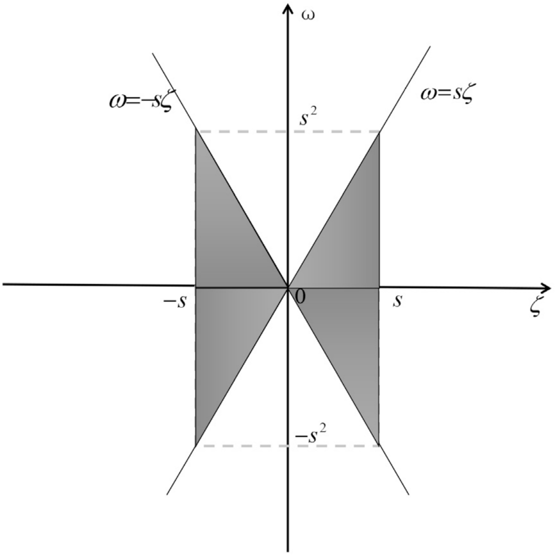



Fig. 3. $E(s)$ is the shaded area, $|\xi|^2 = \zeta^2$.

$$
\begin{aligned}
I(s) &= \int\limits_{E(s)} |\hat{F}(\xi,\omega)|^2 d\xi d\omega \\
&= \int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_0^{s} \Big(\int\limits_{-sr}^{sr} |\hat{F}(r\hat{\xi},\omega)|^2 d\omega\Big) r^2 \sin\varphi dr d\theta d\varphi \\
&\quad + \int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_{-s}^{0} \Big(\int\limits_{sr}^{-sr} |\hat{F}(-r\hat{\xi},\omega)|^2 d\omega\Big) r^2 \sin\varphi dr d\theta d\varphi.
\end{aligned} \tag{5.5}
$$

Let $r = s\hat{r}$ and $\omega = s^2\hat{r}\hat{\omega}$. A simple calculation yields

$$
\begin{aligned}
&\int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_0^{1} \Big(\int\limits_{-s^2\hat{r}}^{s^2\hat{r}} |\hat{F}(s\hat{r}\hat{\xi},\omega)|^2 d\omega\Big) s^3\hat{r}^2 \sin\varphi d\hat{r} d\theta d\varphi \\
&= \int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_0^{1} \Big(\int\limits_{-1}^{1} |\hat{F}(s\hat{r}\hat{\xi}, s^2\hat{r}\hat{\omega})|^2 s^2\hat{r} d\hat{\omega}\Big) s^3\hat{r}^2 \sin\varphi d\hat{r} d\theta d\varphi \\
&= \frac{1}{(2\pi)^4} \int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_0^{1} \Big(\int\limits_{-1}^{1} \Big|\int\limits_{\mathbb{R}^4} F(x,t) e^{-i(s\hat{r}\hat{\xi}\cdot x + s^2\hat{r}\hat{\omega}t)} dx dt\Big|^2 s^2\hat{r} d\hat{\omega}\Big) s^3\hat{r}^2 \sin\varphi d\hat{r} d\theta d\varphi.
\end{aligned}
$$

Similarly, we also obtain the expansion of the second term of (5.5). The integrands $I(s)$ are analytic functions of $s = s_1 + is_2$, $s_1, s_2 \in \mathbb{R}$. Noting that $|e^{-i(s\hat{r}\hat{\xi}\cdot x + s^2\hat{r}\hat{\omega}t)}| \le e^{R|s_2|+2T_0|s_1 s_2|}$, we have for all $s \in S$ that

$$|I(s)| \le CM^2|s|^5 e^{2R|s_2|+4T_0|s_1 s_2|}.$$

Let $m = \max\{2R, 4T_0\}$, it follows for all $s \in S$ that

$$|e^{-(m+1)s-(m+1)s^2} I(s)| \le CM^2.$$

Recalling from (5.4) a prior estimate, we obtain

$$|e^{-(m+1)s-(m+1)s^2} I(s)| \le C\epsilon^2 \quad \text{for all} \quad s \in [0, \Lambda].$$

A direct application of Lemma 4.2 shows that

$$|I(s)| \le CM^2 e^{2(m+1)s}\epsilon^{2\mu} \quad \text{for all} \quad s \in (\Lambda, \infty).$$

Consider the set (see Fig. 4)

$$E_1(s) = \{(\xi, \omega) \in \mathbb{R}^4 |\ \omega^2 - \lambda|\xi|^2 = 0,\ |\xi| \ge s,\ |\omega| \le s|\xi|\}.$$

We have by using the Parseval's identity that

$$I_1(s) := \int\limits_{E_1(s)} |\hat{F}|^2 d\xi\, dw \le \frac{1}{s^2}\int\limits_{\mathbb{R}^3} |\widehat{\nabla F}|^2 d\xi\, dw \le \frac{M^2}{s^2}.$$

Denote (see Fig. 5)

$$E_2(s) = \{(\xi, \omega) \in \mathbb{R}^4 |\ \omega^2 - \lambda|\xi|^2 = 0,\ \ |\omega| \ge s|\xi|\}.$$

Similar to $I(s)$, by using the polar coordinates and $r = s\hat{r}$, $\hat{r} \in (0, 1)$ we get

$$\begin{aligned} I_2(s) &= \int\limits_{E_2(s)} |\widehat{F}(\xi, \omega)|^2 d\xi d\omega \\ &= \int\limits_0^{\infty} \Big(\int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_0^{\frac{1}{s}\omega} |\hat{F}(r\hat{\xi}, \omega)|^2 r^2 \sin\varphi dr\, d\theta\, d\varphi\Big) d\omega \\ &\quad + \int\limits_{-\infty}^{0} \Big(\int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_0^{-\frac{1}{s}\omega} |\hat{F}(r\hat{\xi}, \omega)|^2 r^2 \sin\varphi dr\, d\theta\, d\varphi\Big) d\omega. \end{aligned} \tag{5.6}$$

Simple variable replacement yields

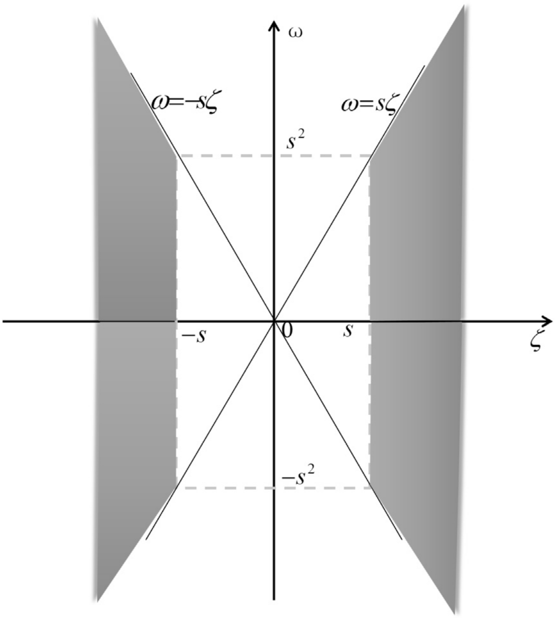



Fig. 4. $E_1(s)$ is the shaded area, $|\xi|^2 = \zeta^2$.

$$\begin{aligned}
&\int_0^\infty \Big(\int_0^{2\pi}\int_0^\pi\int_0^{\frac{1}{s}\omega} |\hat F(r\hat\xi,\omega)|^2 r^2 \sin\varphi\, dr\, d\theta\, d\varphi\Big) d\omega \\
&= \int_0^\infty \Big(\int_0^{2\pi}\int_0^\pi\int_0^{1} |\hat F(\frac{1}{s}\omega\hat r\hat\xi,\omega)|^2 \frac{1}{s^3}\omega^3 \hat r^2 \sin\varphi\, d\hat r\, d\theta\, d\varphi\Big) d\omega \\
&= \frac{1}{s^3}\Big(\int_0^1 + \int_1^\infty \Big(\int_0^{2\pi}\int_0^\pi\int_0^{1} |\hat F(\frac{1}{s}\omega\hat r\hat\xi,\omega)|^2 \omega^3 \hat r^2 \sin\varphi\, d\hat r\, d\theta\, d\varphi\Big) d\omega\Big).
\end{aligned} \tag{5.7}$$

Similarly, we also obtain the expansion of the second term of (5.6). Therefore, combining this with

$$\int_0^1 |\hat F(\frac{1}{s}\omega\hat r\hat\xi,\omega)|^2 d\hat r = \int_0^1 \Big|\frac{1}{(2\pi)^3\omega^3}\int_{\mathbb{R}^3} \partial_t^3 F(x,t) e^{-i(\frac{1}{s}\omega\hat r\hat\xi\cdot x+\omega t)} dx dt\Big|^2 d\hat r,$$

we find

$$|I_2(s)| \le C\,\frac{M^2}{s^3}.$$

**Proof of Theorem 2.3.** We assume that $\epsilon < e^{-1}$. The estimate is obvious if otherwise. Let

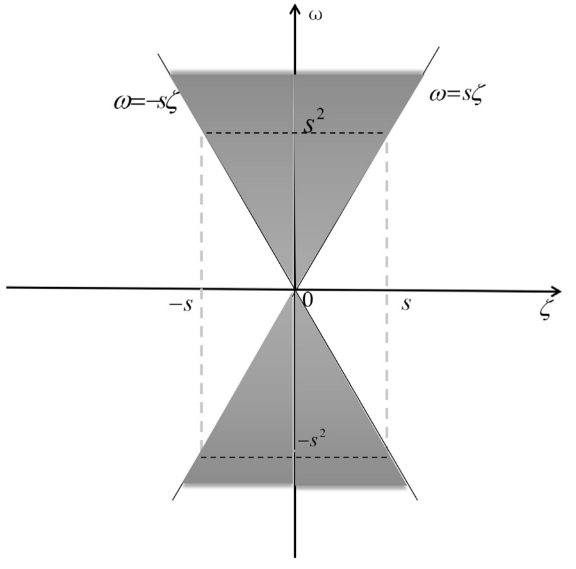



Fig. 5. $E_2(s)$ is the shaded area, $|\xi|^2 = \zeta^2$.

$$s = \begin{cases} \frac{1}{(2(m+2)\pi)^{\frac{1}{4}}} \Lambda^{\frac{1}{2}} |\ln \epsilon|^{\frac{1}{4}(1-\alpha)}, & \text{if } |\ln \epsilon|^{\frac{1}{4}(1-\alpha)} > 2^{\frac{1}{4}} \Lambda^{\frac{1}{2}} (2(m+2)\pi)^{\frac{1}{4}}, \\ \Lambda, & \text{if } |\ln \epsilon|^{\frac{1}{4}(1-\alpha)} \le 2^{\frac{1}{4}} \Lambda^{\frac{1}{2}} (2(m+2)\pi)^{\frac{1}{3}}. \end{cases} \tag{5.8}$$

Here $\alpha \in (0, 1)$ is a constant.

Case (i): $|\ln \epsilon|^{\frac{1}{4}(1-\alpha)} > 2^{\frac{1}{4}} \Lambda^{\frac{1}{2}} (2(m+2)\pi)^{\frac{1}{4}}$. One can check that

$$s > 2^{\frac{1}{4}} \Lambda.$$

Thus, using Lemma 4.2, we obtain

$$\begin{aligned} |I(s)| &\le C\, e^{2(m+2)s^2} \epsilon^{2\mu(s)} \\ &\le C e^{-2\mu(s)|\ln \epsilon| + 2(m+2)s^2} \\ &\le C e^{-(\frac{-2|\ln \epsilon|}{\pi}(\frac{\Lambda}{s})^2 + 2(m+2)s^2)} \\ &\le C e^{-2(\frac{2(\Delta+2)}{\pi})^{\frac{1}{2}} \Lambda |\ln \epsilon|^{1-\frac{1}{2}(1-\alpha)} (1 - \frac{1}{2}|\ln \epsilon|^{-\alpha})}. \end{aligned} \tag{5.9}$$

Noting that $\frac{1}{2}|\ln \epsilon|^{-\alpha} < \frac{1}{2}$ and $(\frac{2(m+2)}{\pi})^{\frac{1}{2}} > 1$, we have

$$|I(s)| \le C e^{-\Lambda |\ln \epsilon|^{1-\frac{1}{2}(1-\alpha)}}.$$

Using the elementary inequality

$$e^{-t} \le \frac{1}{t}, \quad t > 0,$$

we get

$$|I(s)| \le C \frac{M^2}{\left(\Lambda |\ln\epsilon|^{1-\frac{1}{2}(1-\alpha)}\right)}.$$

Observing that $\mathbb{R}^4 = E(s) \cup E_1(s) \cup E_2(s)$, we know

$$\begin{aligned}
\|F\|^2_{L^2(\mathbb{R}^4)} = \|\hat{F}\|^2_{L^2(\mathbb{R}^4)} &= \int\limits_{E(s)} |\hat{F}|^2 d\xi\, d\omega + \int\limits_{E_1(s)} |\hat{F}|^2 d\xi\, d\omega + \int\limits_{E_2(s)} |\hat{F}|^2 d\xi\, d\omega \\
&= I(s) + I_1(s) + I_2(s) \\
&\le C\Big(\frac{M^2}{\Lambda |\ln\epsilon|^{1-\frac{1}{2}(1-\alpha)}} + \frac{M^2}{\Lambda |\ln\epsilon|^{\frac{1}{2}(1-\alpha)}}\Big).
\end{aligned}$$

Since $\left(\Lambda |\ln\epsilon|^{\frac{1}{2}(1-\alpha)}\right) < \left(\Lambda |\ln\epsilon|^{1-\frac{1}{2}(1-\alpha)}\right)$ when $\Lambda > 1$, $|\ln\epsilon| > 1$ and $0 < \alpha < 1$, we obtain

$$\|F\|^2_{L^2(\mathbb{R}^4)} \le C \frac{M^2}{\Lambda |\ln\epsilon|^{\frac{1}{2}(1-\alpha)}}. \tag{5.10}$$

Case (ii): $|\ln\epsilon|^{\frac{1}{4}(1-\alpha)} \le 2^{\frac{1}{4}} \Lambda^{\frac{1}{2}} (2(\Delta+2)\pi)^{\frac{1}{4}}$. In this case we have for $s = \Lambda$ that

$$|I(s)| = |I(\Lambda)| \le |E(\Lambda)| \Lambda^4 (1+\Lambda)^2 \epsilon^2.$$

Using estimates of $I_1(s)$ and $I_2(s)$, we obtain

$$\begin{aligned}
\|F\|^2_{L^2(\mathbb{R}^4)} =& \|\hat{F}\|^2_{L^2(\mathbb{R}^4)} \\
=& I(s) + I_1(s) + I_2(s) \\
\le& C\Big(|E(\Lambda)| \Lambda^4 (1+\Lambda)^2 \epsilon^2 + \frac{M^2}{\Lambda |\ln\epsilon|^{\frac{1}{2}(1-\alpha)}}\Big).
\end{aligned} \tag{5.11}$$

Combining (5.10) and (5.11), we finally derive

$$\|F\|^2_{L^2(\mathbb{R}^4)} \le C\Big(\Lambda^{10} \epsilon^2 + \frac{M^2}{\Lambda |\ln\epsilon|^{\frac{1}{2}(1-\alpha)}}\Big).$$

This completes the proof. □

## 6. Proof of Theorem 2.4

In this section, we assume that $F$ takes the form

$$F(x_1, x_2, x_3, t) = f(\tilde{x}, t) g(x_3), \quad \tilde{x} = (x_1, x_2) \in \mathbb{R}^2, \quad t \in (0, +\infty). \tag{6.12}$$

Then the equation (1.1) becomes

$$\partial_t^2 u - \Delta u = f(\tilde{x}, t)\, g(x_3). \tag{6.13}$$

Assuming that $g$ is known, we establish an increasing stability estimate for $f$ from the Dirichlet data $\{u(x,t)|\ x \in \partial B_R,\ t \in (0,T)\}$. We choose the test function

$$w(\xi,t) = e^{-i(\xi_1 \cdot x_1 + \xi_2 \cdot x_2 + \xi_3 x_3 + \omega t)}, \quad \omega^2 - |\xi_1|^2 - |\xi_2|^2 = |\xi_3|^2,$$

which satisfies the wave equation

$$\partial_t^2 w - \Delta w = 0.$$

By the Huygens' principle, it holds that $u(x,t) = 0$ for $|x| < R$ and $t > T$. Multiplying $w$ on both sides of (6.13) and integrating over $B_R \times (0,T)$, we have

$$\begin{aligned}(2\pi)^2 \hat{f}(\xi_1,\xi_2,\omega)\hat{g}(\xi_3) &= \int_{\mathbb{R}^3} f(\tilde{x},t) e^{-i(\tilde{\xi}\cdot\tilde{x}+\omega t)} d\tilde{x}dt \int_{\mathbb{R}} g(x_3) e^{-i\xi_3 \cdot x_3} dx_3 \\ &= \int_0^T \int_{\partial B_R} (u\frac{\partial w}{\partial \nu} - w\frac{\partial u}{\partial \nu}) ds(x)dt,\end{aligned} \tag{6.14}$$

where $\tilde{\xi} = (\xi_1,\xi_2)$. Consider the set

$$E(b) = \{(\xi_1,\xi_2,\omega) \in \mathbb{R}^3 |\ \omega^2 - |\xi_1|^2 - |\xi_2|^2 = |\xi_3|^2,\ \xi_3 \in (-b,b),\ |\xi_1|^2 + |\xi_2|^2 \le b^2\}.$$

Since $\xi_3 \in (-b,b)$, one can check that $|E(b)| > 0$. Using the fact that $f$ and $g$ have compact supports and arguing analogously to Lemma 3.2, we deduce

$$\|\partial_\nu u\|_{L^2([0,T];H^{\frac{1}{2}}(\partial B_R))} \leqslant C\|u\|_{H^2([0,T];H^{\frac{3}{2}}(\partial B_R))}. \tag{6.15}$$

Combining (6.15) and $|\hat{g}(\xi_3)| \ge \delta > 0$ for $\xi_3 \in (-b,b)$ with (6.14), we get for $(\xi_1,\xi_2,\omega) \in E(b)$ that

$$|\hat{f}(\xi_1,\xi_2,\omega)| \le C\frac{(1+|\xi|)\|u\|_{H^2([0,T];H^{\frac{3}{2}}(\partial B_R))}}{|\hat{g}(\xi_3)|} \le C\delta^{-1}(1+|\xi|)\epsilon, \tag{6.16}$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$.

Define the set (see Fig. 6)

$$E(s) = \{(\xi_1,\xi_2,\omega) \in \mathbb{R}^3 |\ \omega^2 - |\xi_1|^2 - |\xi_2|^2 = |\xi_3|^2,\ \xi_3 \in (-s,s),\ |\xi_1|^2 + |\xi_2|^2 \le s^2\}.$$

Let

$$I(s) = \int_{E(s)} |\hat{f}(\xi_1,\xi_2,\omega)|^2 d\xi_1 d\xi_2 d\omega.$$

Using the polar coordinates $\xi_1 = r\sin\theta$, $\xi_2 = r\cos\theta$, $0 \le r \le s$, $0 \le \theta \le 2\pi$, we obtain that

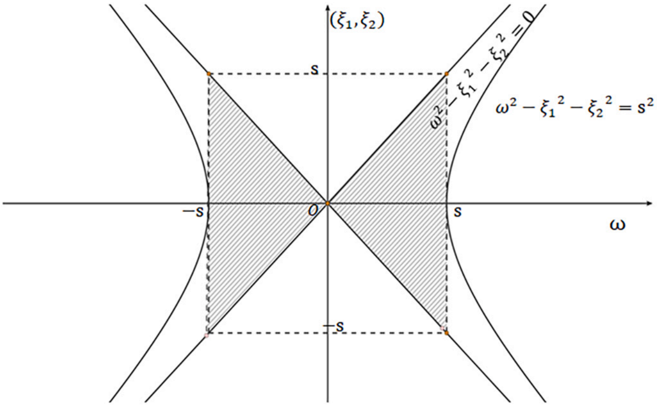



Fig. 6. $E(s)$ is the shaded area.

$$I(s)=\int_{-s}^{0}\int_{0}^{2\pi}\int_{0}^{-\omega}|\hat{f}(-r\sin\theta,-r\cos\theta,\omega)|^2(-r)drd\theta d\omega$$
$$+\int_{0}^{s}\int_{0}^{2\pi}\int_{0}^{\omega}|\hat{f}(r\sin\theta,r\cos\theta,\omega)|^2rdrd\theta d\omega.$$

Let $\omega=s\overline{\omega}$ for $\overline{\omega}\in(-1,1)$ and $r=s\overline{r}$ for $\overline{r}\in(0,1)$. Then a simple calculation yields

$$I(s)=\int_{-1}^{0}\int_{0}^{2\pi}\int_{0}^{-\overline{\omega}}|\hat{f}(-s\overline{r}\sin\theta,-s\overline{r}\cos\theta,s\overline{\omega})|^2s^3(-\overline{r})d\overline{r}d\theta d\overline{\omega}$$
$$+\int_{0}^{1}\int_{0}^{2\pi}\int_{0}^{\overline{\omega}}|\hat{f}(s\overline{r}\sin\theta,s\overline{r}\cos\theta,s\overline{\omega})|^2s^3\overline{r}d\overline{r}d\theta d\overline{\omega}.$$

This integral $I(s)$ is also an analytic function of $s=s_1+is_2\in\mathbb{C}$. Noting that $|e^{-i(\pm s\overline{r}(\sin\theta,\cos\theta)\cdot\tilde{x}+s\overline{\omega}t)}|\le e^{(R_0+T_0)|s_2|}$ for all $\tilde{x}\in\tilde{B}_{R_0}$ and $t\in(0,T_0)$, we deduce

$$\begin{aligned}(2\pi)^{\frac{3}{2}}|\hat{f}(\pm s\overline{r}\sin\theta,s\overline{r}\cos\theta,s\overline{\omega})|^2&=|\int_{0}^{T_0}\int_{\tilde{B}_{R_0}}f(\tilde{x},t)e^{-i(\pm s\overline{r}(sin\theta,cos\theta)\cdot\tilde{x}+s\overline{\omega}t)}d\overline{x}dt|^2\\&\le Ce^{2(R_0+T_0)|s_2|}\|f\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}\tag{6.17}$$

Thus

$$|I(s)|\le C|s|^3e^{2(R_0+T_0)|s_2|}\|f\|^2_{L^2(\mathbb{R}^3)},$$

which yields

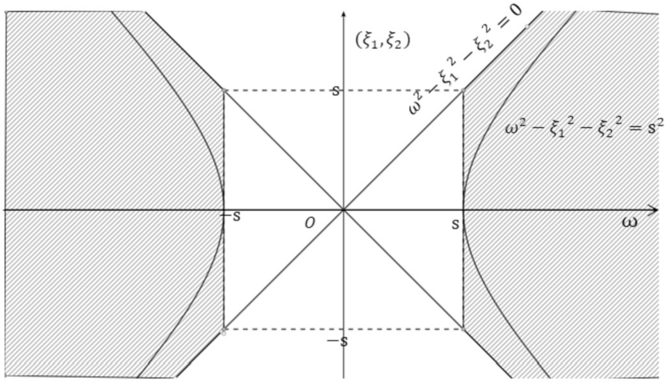



Fig. 7. $E_1(s)$ is the shaded area.

$$|I(s)e^{-(2(R_0+T_0)+1)s}| \le CM^2 \text{ for all } s \in S.$$

Together with the estimate (6.16) and definition of $E(s)$, we have

$$|I(s)| \le C\,|E(s)|\,(1+s)^2\,\epsilon^2 \quad \text{for all} \quad s \in [0,b]. \tag{6.18}$$

Applying Lemma 4.2, we know for all $s > b$ that

$$|I(s)| \le CM^2 e^{(2(R_0+T_0)+1)s}\epsilon^{2\mu(s)}.$$

Define $E_1(s) := \{(\xi_1,\xi_2,\omega) \in \mathbb{R}^3 \,|\, |\omega| > s,\ \xi_1^2+\xi_2^2 \le \omega^2\}$ (see Fig. 7) and

$$I_1(s) = \int\limits_{E_1(s)} |\hat{f}(\xi_1,\xi_2,\omega)|^2 d\xi_1 d\xi_2 d\omega.$$

Since $\operatorname{supp} f(\tilde{x},t) \subset \widetilde{B}_{R_0} \times (0,T_0)$, we have

$$\begin{aligned}
(2\pi)^{\frac{3}{2}}\hat{f}(\xi_1,\xi_2,\omega) &= \int\limits_0^{T_0}\int\limits_{\tilde{B}_{R_0}} f(x_1,x_2,t)e^{-i(\xi_1\cdot x_1+\xi_2\cdot x_2+\omega t)}dx_1dx_2dt \\
&= \frac{1}{-i\omega}\int\limits_0^{T_0}\int\limits_{\tilde{B}_{R_0}} \partial_t f(x_1,x_2,t)e^{-i(\xi_1\cdot x_1+\xi_2\cdot x_2+\omega t)}dx_1dx_2dt \\
&= \frac{1}{-i\omega}\widehat{\partial_t f}.
\end{aligned}$$

Then using the Parseval's identity, it's easy to get

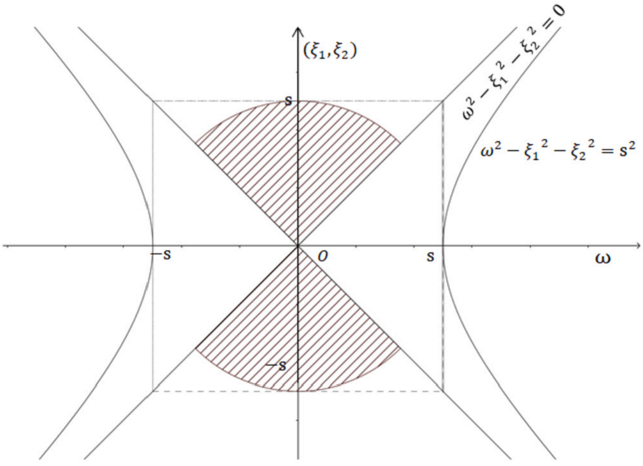



Fig. 8. $E_2(s)$ is the shaded area.

$$I_1(s) = \int_{E_1(s)} |\widehat{f}(\xi_1,\xi_2,\omega)|^2 d\xi_1 d\xi_2 d\omega \le \frac{1}{s^2} \int_{E_1(s)} |\widehat{\nabla f}(\xi_1,\xi_2,\omega)|^2 d\xi_1 d\xi_2 d\omega \le \frac{CM^2}{s^2}. \quad (6.19)$$

Define the set (see Fig. 8)

$$E_2(s) := \{(\xi_1,\xi_2,\omega) \in \mathbb{R}^3 | \, \omega^2 + |\xi_1|^2 + |\xi_2|^2 \le s^2, \ \xi_1^2 + \xi_2^2 \ge \omega^2\}.$$

Now we estimate

$$I_2(s) = \int_{E_2(s)} |\hat{f}(\xi_1,\xi_2,\omega)|^2 d\xi_1 d\xi_2 d\omega.$$

Define $F_b(\xi_1,\xi_2,\omega) = \hat{f}(b\xi_1, b\xi_2, b\omega)$ for $(\xi_1,\xi_2,\omega) \in \mathbb{R}^3$. Since $f$ is compactly supported, one can see that the function $F_b$ is analytic and satisfies for $\gamma \in (\mathbb{N} \cup \{0\})^3$ that

$$\begin{aligned}
|\partial^\gamma F_b(\xi_1,\xi_2,\omega)| &= |\partial^\gamma F(b\xi_1, b\xi_2, b\omega)| \\
&= \Big| \partial^\gamma \int_{\mathbb{R}^3} f(x_1,x_2,t) e^{-ib(\xi_1\cdot x_1 + \xi_2 \cdot x_2 + \omega t)} dx_1 dx_2 dt \Big| \\
&= \Big| \sum_{\substack{\alpha+\beta=\gamma \\ \alpha,\beta\in\mathbb{N}\cup 0}} \int_{\mathbb{R}^3} (-i)^{|\gamma|} b^{|\gamma|} x^\alpha t^\beta f(x_1,x_2,t) e^{-ib(\xi_1\cdot x_1 + \xi_2 \cdot x_2 + \omega t)} dx_1 dx_2 dt \Big|.
\end{aligned}$$

Using $b^{|\gamma|} < |\gamma|! \, e^b$ and $\eta = \max\{R, T\}^{-1}$, we obtain

$$|\partial^\gamma F_b(\xi_1,\xi_2,\omega)| \le \|f\|^2_{L^2(B_R)} \eta^{-|\gamma|} b^{|\gamma|} \le C \|f\|^2_{L^2(B_R)} \eta^{-|\gamma|} |\gamma|! \, e^b \le CM^2 \eta^{-|\gamma|} |\gamma|! \, e^b. \quad (6.20)$$

Applying Proposition 3.3 to the set $\mathcal{O} := E(1)$, we can find a constant $\alpha \in (0, 1)$ such that

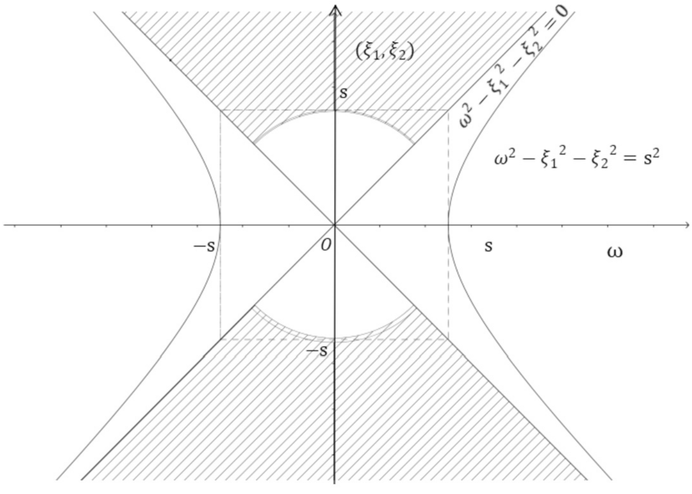



Fig. 9. $E_3(s)$ is the shaded area.

$$\|F_b\|_{L^\infty(B_1)} \leq C\, e^{b(1-\alpha)} \|F_b\|^\alpha_{L^\infty(\mathcal{O})}.$$

Using the fact that $\hat{f}(\xi) = F_b(b^{-1}\xi)$, one gets the following estimate

$$\begin{aligned}\|\hat{f}\|_{L^\infty(B(0,b))} &= \|F_b\|_{L^\infty(B_1)} \leq C e^{b(1-\alpha)} \|F_b\|^\alpha_{L^\infty(\mathcal{O})} \\ &\leq C e^{b(1-\alpha)} \|\hat{f}\|^\alpha_{L^\infty(E(b))} \leq C e^{b(1-\alpha)} (1+b)^\alpha \epsilon^\alpha. \end{aligned} \tag{6.21}$$

Combining (6.16), we know for $s \in [0, b]$ that

$$|I_2(s)| \leq C |E_2(b)|\, e^{2b(1-\alpha)}\, (1+b)^{2\alpha}\, \epsilon^{2\alpha}$$

and similarly

$$|I_2(s)| \leq C M^2\, e^{(2(R_0+T_0)+1)|s_2|} \quad \text{for all} \quad s \in S.$$

Thus applying Lemma 4.2, we have

$$|e^{-[2(R_0+T_0)+1]s} I_2(s)| \leq C M^2 (\epsilon^\alpha)^{2\mu} \quad \text{for all} \quad s > b.$$

Consider the set (see Fig. 9)

$$E_3(s) := \{(\xi_1, \xi_2, \omega) \in \mathbb{R}^3 |\ \omega^2 + |\xi_1|^2 + |\xi_2|^2 > s^2,\ \xi_1^2 + \xi_2^2 \geq \omega^2\}.$$

As done in the estimate of $I_1(s)$, applying Parseval's identity yields

$$I_3(s) = \int\limits_{E_3(s)} |\hat{f}(\xi_1, \xi_2, \omega)|^2 d\xi_1 d\xi_2 d\omega \leq \frac{CM^2}{s^2}. \tag{6.22}$$

**Proof of Theorem 2.4.** We assume that $\epsilon < e^{-1}$, since if otherwise the estimate is obvious. Let

$$s = \begin{cases} \frac{1}{((2(R_0+T_0)+3)\pi)^{\frac{1}{3}}} b^{\frac{2}{3}} |\ln\epsilon|^{\frac{1}{4}}, & \text{if } |\ln\epsilon|^{\frac{1}{4}} > 2^{\frac{1}{4}} b^{\frac{1}{3}} ((2(R_0+T_0)+3)\pi)^{\frac{1}{3}}, \\ b, & \text{if } |\ln\epsilon|^{\frac{1}{4}} \le 2^{\frac{1}{4}} b^{\frac{1}{3}} ((2(R_0+T_0)+3)\pi)^{\frac{1}{3}}. \end{cases} \tag{6.23}$$

Case (i): $|\ln\epsilon|^{\frac{1}{4}} > 2^{\frac{1}{4}} b^{\frac{1}{3}} ((2(R_0+T_0)+3)\pi)^{\frac{1}{3}}$. One can check that

$$s > 2^{\frac{1}{4}} b.$$

Thus, using Lemma 4.2 we obtain

$$\begin{aligned} |I(s)| &\le CM^2 e^{(2(R_0+T_0)+3)s} \epsilon^{2\mu} \\ &\le Ce^{-2\mu|\ln\epsilon| + ((2(R_0+T_0)+3)s} \\ &\le CM^2 e^{-\frac{-2|\ln\epsilon|}{\pi}(\frac{b}{s})^2 + \frac{(2(R_0+T_0)+3)}{((2(R_0+T_0)+3)\pi)^{\frac{1}{3}}} b^{\frac{2}{3}} |\ln\epsilon|^{\frac{1}{4}}} \\ &\le CM^2 e^{-2(\frac{(2(R_0+T_0)+3)^2}{\pi})^{\frac{1}{3}} b^{\frac{2}{3}} |\ln\epsilon|^{\frac{1}{2}} (1-\frac{1}{2}|\ln\epsilon|^{-\frac{1}{4}})}. \end{aligned} \tag{6.24}$$

Noting that $\frac{1}{2}|\ln\epsilon|^{-\frac{1}{4}} < \frac{1}{2}$ and $\left(\frac{(2(R_0+T_0)+3)^2}{\pi}\right)^{\frac{1}{3}} > 1$, we have

$$|I(s)| \le CM^2 e^{-b^{\frac{2}{3}} |\ln\epsilon|^{\frac{1}{2}}}.$$

Using the elementary inequality

$$e^{-t} \le \frac{C}{t^2}, \quad t > 0,$$

we get

$$|I(s)| \le \frac{CM^2}{\left(b^{\frac{2}{3}} |\ln\epsilon|^{\frac{1}{2}}\right)^2}.$$

Similarly, we have

$$|I_2(s)| \le \frac{CM^2}{\left(b^{\frac{2}{3}} |\alpha \ln\epsilon|^{\frac{1}{2}}\right)^2}.$$

Thus

$$
\begin{aligned}
\|f\|^2_{L^2(\mathbb{R}^3)} &= \|\hat{f}\|^2_{L^2(\mathbb{R}^3)} \\
&= \int\limits_{E(s)} |\hat{f}(\tilde{\xi},\omega)|^2 \, d\tilde{\xi}\, d\omega + \int\limits_{E_1(s)} |\hat{f}(\tilde{\xi},\omega)|^2 \, d\tilde{\xi}\, d\omega \\
&\quad + \int\limits_{E_2(s)} |\hat{f}(\tilde{\xi},\omega)|^2 \, d\tilde{\xi}\, d\omega + \int\limits_{E_3(s)} |\hat{f}(\tilde{\xi},\omega)|^2 \, d\tilde{\xi}\, d\omega \\
&= I(s) + I_1(s) + I_2(s) + I_3(s) \\
&\le C\Big(\frac{M^2}{\big(b^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{2}}\big)^2} + \frac{M^2}{\big(b^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{4}}\big)^2} + \frac{M^2}{\big(b^{\frac{2}{3}}|\alpha\ln\epsilon|^{\frac{1}{2}}\big)^2} + \frac{M^2}{\big(b^{\frac{2}{3}}|\alpha\ln\epsilon|^{\frac{1}{4}}\big)^2}\Big).
\end{aligned}
$$

Since $b^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{2}} > b^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{4}}$, $b^{\frac{2}{3}}|\alpha\ln\epsilon|^{\frac{1}{2}} > b^{\frac{2}{3}}|\alpha\ln\epsilon|^{\frac{1}{4}}$ and $b^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{4}} > b^{\frac{2}{3}}|\alpha\ln\epsilon|^{\frac{1}{4}}$ when $b > 1$ and $|\ln\epsilon| > 1$, we obtain

$$
\|f\|^2_{L^2(\mathbb{R}^3)} \le C\frac{M^2}{\big(b^{\frac{2}{3}}|\alpha\ln\epsilon|^{\frac{1}{4}}\big)^2}. \tag{6.25}
$$

Case (ii): $|\ln\epsilon|^{\frac{1}{4}} > 2^{\frac{1}{4}} b^{\frac{1}{3}}((2(R_0+T_0)+3)\pi)^{\frac{1}{3}}$. In this case we have from $s = b$ and (6.18) that

$$
|I(s)| = |I(b)| \le |E(b)|(1+b)^2\epsilon^2.
$$

Combining this estimate and $I_1(s)$, we obtain

$$
\begin{aligned}
\int\limits_{E(s)} |\hat{f}(\tilde{\xi},\omega)|^2 \, d\tilde{\xi}\, d\omega + \int\limits_{E_1(s)} |\hat{f}(\tilde{\xi},\omega)|^2 \, d\tilde{\xi}\, d\omega &= I(s) + I_1(s) \\
&\le C\Big(b^5\epsilon^2 + \frac{M^2}{\big(b^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{4}}\big)^2}\Big).
\end{aligned} \tag{6.26}
$$

Similarly, we have

$$
|I_2(s)| = |I_2(b)| \le |E_2(b)| e^{2b(1-\alpha)}(1+b)^{2\alpha}\epsilon^{2\alpha}
$$

and

$$
\begin{aligned}
\int\limits_{E_2(s)} |\hat{f}(\tilde{\xi},\omega)|^2 \, d\tilde{\xi}\, d\omega + \int\limits_{E_3(s)} |\hat{f}(\tilde{\xi},\omega)|^2 \, d\tilde{\xi}\, d\omega &= I_2(s) + I_3(s) \\
&\le C\Big(b^3\, e^{2b(1-\alpha)}(1+b)^{2\alpha}\epsilon^{2\alpha} + \frac{M^2}{\big(b^{\frac{2}{3}}|\alpha\ln\epsilon|^{\frac{1}{4}}\big)^2}\Big).
\end{aligned} \tag{6.27}
$$

Combining (6.25), (6.26) and (6.27), we obtain

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{R}^3)} &= \|\widehat{f}\|^2_{L^2(\mathbb{R}^3)} \\ &= I(s) + I_1(s) + I_2(s) + I_3(s) \\ &\le C\Big(b^5\epsilon^2 + b^3\ e^{2b(1-\alpha)}(1+b)^{2\alpha}\epsilon^{2\alpha} + \frac{M^2}{\big(b^{\frac{2}{3}}|\alpha\ln\epsilon|^{\frac{1}{4}}\big)^2}\Big).\end{aligned}$$

This completes the proof. □

## 7. Conclusion

In this work, we have shown increasing stability estimates of the $L^2$-norm of the source function with respect to interval length of a given parameter. In the first inverse problem (IP1), this parameter was chosen as the frequency bandwith of the temporal component of the source function. We are mostly motivated by the increasing analysis for time-harmonic inverse source problems, and other type of increasing stability estimates for the wave equation might be possible. Our results are expected to be valid in the two-dimensional case and also for more general evolutional equations. In three dimensions, the measurement data are taken over a large time interval due to Huygens' principle. The absence of Huygens' principle in 2D will create technical difficulties using a finite time-interval data and a new technique must be developed. Although zero initial conditions for the wave equation are considered in this work, the mathematical analysis performed here carries over to non-zero initial conditions naturally. Inverse source problems with non-zero conditions have appeared in for instance thermoacoustic and photoacoustic tomography (TAT/PAT), which are indeed an important research topic. We hope to be able to report the progress on these problems in the future.

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## Data availability

No data was used for the research described in the article.

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